# SCIENCE AND MATHEMATICS

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That science has been dependent upon mathematics in many ways is so well recognized that it needs no comment. Sooner or later each branch of science develops a system of observational methods which assume numerical counting or the registration of numerical results of measure. These statistical results or quantitative results must then be discussed and their significance ascertained. It is true, of course, that there are other mathematical conceptions than those of numbers which enter largely into scientific work, but these cannot be discussed in the present paper. Such conceptions as vector, vectorline, dyadic, fields, whether scalar vector or dyadic, curl, divergence, line-integral, may involve numerical elements, but the essence of their characters is non-numerical. These, however, must be passed by in order to remain within the limits of time, and I desire to consider only one notion that science owes to mathematics and which appears in one way or another in practically every science.

The notion of scientific law rests upon the mathematical concept of functionality. In a law it is stated that a certain effect is to be expected from a given cause, or to be rather more technical and at the same time more exact, that a certain phenomenon called the effect is a function of a certain phenomenon called the cause. If a quantitative measure can be applied to these two phenomena the law may then be stated in a formula of the type $y=f(x)$. We may leave to one side for the sake of definiteness the functions that depend upon more than one variable. The problem then in determining an exact law is that of ascertaining the character of $f$.

Now when we define a function we must by the definition be able to calculate in some manner the value of the dependent variable $y$, for any assigned value of the independent variable $x$. If the independent variable can assume only a finite and in fact a relatively small number of values, then our mode of ascertaining $y$ may be reference to a table of values giving $y$ for each $x$, as for example the farmer refers to the almanac to find the time of sunrise for each day of the year. If the number of values necessary to consider becomes relatively large, though still finite, as, for instance, $10^{24}$, the limitations of humanity make it impracticable to utilize a table of values to ascertain $y$ when $x$ is given. In this case and in the case in which there is an infinity of values of $x$, the values of $y$ must be given by some kind of an expression which can be computed in at least a reasonable time.

In determining the laws of science, we find that in some cases these are worked out from a given set of observations, necessarily finite in number, and indeed relatively few. In order to determine a general formula, then, which will hold good for an infinity of cases or for a relatively large number of cases, it becomes necessary to supplement the observation with various hypotheses, or assumptions. The most common of these is the assumption of continuity, which means that if we change the value of the independent variable by a variable increment, the change in $y$ will be a variable increment (including the case when this variable assumes equal values for all its range) and the two increments decrease together, indefinitely. That this assumption is the most natural one for the mind to make would be quite evident if we accept C. S. Peirce's analysis of mind, in which he finds the great characteristic of mind is its continuity, which, indeed, is Bergson's conclusion. In any case it seemed for a long time that if for every value of $x$ between two given values, $x_1$ and $x_2$, $y$ must assume every value intermediate between $y_1$ corresponding to $x$, and $y_2$ corresponding to $x_2$, then $y$ would have to be a continuous function of $x$. But in the progress of mathematics Darboux invented a function which does assume between $y_1$ and $y_2$ every intermediate value and yet is discontinuous everywhere. The significance of this invention for science is that science is no longer compelled to assume continuity in order to have the property cited. It is no longer necessary to depend upon actual continuity of values, that is to say, states may change instantaneously by finite amounts and yet assume every value between two given states.

Another common assumption is that of derivative. In many investigations it is assumed that if we are concerned with the ratio of two increments which decrease together, we must substitute for the limit of the ratio a derivative. For example, it is assumed that if we divide a distance-interval of a moving point by the corresponding time-interval, giving an average velocity, that in all actual motions such a ratio must, as the intervals are decreased, have a limiting value called the instantaneous velocity. So long as all phenomena of motion were supposed to be continuous this seemed to be a necessary assumption. But we know now that such a thing as instantaneous velocity may not exist at all. For Weierstrass invented a continuous function which does not have a derivative anywhere, and since his time a great many others have been invented. There is no reason at all why change of position may not take place continuously and yet with no definable velocity anywhere, although there could be an average velocity over any interval which for a smaller interval might run up as near $\infty$ as we like. If we remember the fact that a point which changes its position in time may assume two given values of $x$, and during the moment between two given instants of time, may assume every value between $x_1$ and $x_2$, and do this discontinuously, and the fact that it may do the same thing continuously and yet with no definable velocity at each intermediate instant of time, we certainly offer the scientists a chance to accept even radical modern atomistic conceptions of matter, eleccticity and energy, and yet not pass outside the range of the definable function. If he were to do this, however, he would be dependent entirely upon the mathematician for the development of such laws and the study of their consequences. That the phenomena of nature do not actually take place in this way we have no right to assert. While we have deduced many laws on the hypothesis of differentiability as a basis, we must recognize that it is not a necessary hypothesis. We may assert confidently that, for example, there may be motions in which there is a change of place, a definable velocity, and yet no definable acceleration, even though there may be a field of force in which the particle moves. What becomes of Newton's second law in such case? Again the ordinary laws of dynamics are based upon the assumption that there are practically no shocks or collisions, but if we suppose that there is a relatively very large number of collisions or an infinite number, then the solutions of the actual dynamic laws would have to be functions with a relatively dense set of discontinuities, or even infinitely discontinuous functions. That such are possible every mathematician knows.

We may go still further, however. Many laws of science are the solutions of either total or partial differential equations. Now it is generally supposed that if we have the second derivative of a variable given in an equation, we may from this equation find the third derivative, then the fourth, etc., and thus for a domain not too large find the coefficients of a Taylor series for the function. That is to say more briefly we find that most functions are assumed to be analytic. Now it is well known that in a wide field of physics these analytic solutions are, even if possible, of no interest, and that we must seek for solutions which are at least nonanalytic on certain boundaries. But we can still advance owing to the investigations of Borel, for he invented a differentiable function which in no region, however small, is analytic. This function, for example, would permit us to study the potential in a region which was discontinuous at a set of points everywhere as dense as rational numbers. We do not then have to assume that if a function is continuous, and it has derivatives that it is necessarily analytic.

Turning now to another class of investigations, we find that it is generally assumed in some sciences that the succession of phenomena depend only upon those phenomena that immediately precede. Indeed, this condition permits the use of calculus. But the assumption is purely gratuitous, for it is possible to devise functions in which not only the preceding state but other preceding states act upon the present. It is as if the remote past can reach out a ghostly hand and affect the present. The bearing such a possibility has upon heredity is evident at once. And the mathematician, since Volterra devised such functions, can assert that the assumption of no action over an interval of time is simply an assumption.

A still different kind of function is also due to Volterra and others, namely the function dependent upon a whole infinity of independent variables. Usually the number of independent variables is assumed to be relatively small. The whole tendency of science is to reduce the number of causes. This may be due to the fact that functions of an infinity of variables had never been studied. But this obstacle is now removed and in the study of functions of lines, surfaces, etc., as well as various functional spaces, we have the development of a means of statement of law which may be more useful in the future than it is now, as it is extended to wider fields of science.

Time does not permit more than a mention of the far-reaching dependence upon the laws of statistical mathematics that science has immediately at hand. Problems of distribution, organized and unorganized; problems of large numbers, problems of values that are given only as averages, problems of functions that merely must keep their values within a given approximation, all these suggest that the conception even that a measurement actually has a unique value which could be determined exactly is also an assumption.

But I hear the objection raised, that the fact that the mathematician is able to amuse himself with creations of this kind is one thing, and that the laws of nature belong to their applications is another. I need only remind you, however, that conic sections were studied many centuries before Kepler lived, and that wireless telegraphy is very dependent upon the square root of minus one. This order of events does not always occur but it happens often enough to answer the objection.

In the effort to explain the universe, science is driven more and more toward the postulate that the universe is infinitely complex, and away from the postulate that the universe is comparatively simple. The intricacies of phenomena increase year by year, and the scientist, like the mathematician, is compelled to admit that to generalize merely by adding more terms is a very poor way to generalize. The generalization necessary to handle nature, like the generalization necessary to handle increasing knowledge of mathematics, is plainly a generalization of kind, that kind of generalization which will exhibit the simple case as a degenerate form of the usual case. A more profound insight into the tangle of phenomena shows that the threads are not simple and well known curves merely mixed together, but that they are in reality infinitely complex curves intertwined with themselves. That some of them from certain viewpoints project into straight lines, or circles, or simple helices, is purely an accident. Since this is the case it becomes plainly evident that progress in science is very dependent upon the creative power of the mathematician in matching intricacy in nature with intricacy in mental construction. It becomes very plain that such heavy assumptions as continuity, vicinal action, analyticity, uniqueness, determinability, and others of the same ponderous character, must be left aside, and that discontinuity, distal action, monogeneity, polydromicity, statisticity, and the like, must become the more frequent. And all these demand mathematical development which is as yet only dreamed of.