ON MULTIPLY MUTANT SETS

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In this paper we will extend and develop some of the results of two earlier papers (Mullin, 1960, 1961). First we consider non-empty sets on which there is defined some non-empty index set of closed *binary* composition laws. Secondly, we show some concrete and general properties of sets that satisfy an anti-closure condition relative to all of the elements of some non-empty subset of the index set of composition laws.

In the additive and multiplicative monoids of non-negative integers the set of all odd primes is a doubly mutant set, i.e., it is a mutant with respect to both addition and multiplication.

RESULTS

Prelemma: Consider the algebraic system determined by the additive and multiplicative monoids of non-negative integers. Then there exists, *in a constructive sense*, a maximal doubly mutant set of that system.

Proof: Put E equal to the set of all odd primes, i.e., $\{3,5,7\ldots\}$. Consider the following infinite sequence A_j of infinite sets:

$$A_0 = E,\ A_1 = \{P_{i_1}P_{i_2}P_{i_3} : P_{i_j} \in E\},\ \ldots,$$

$$A_n = \{P_{i_1}P_{i_2}\ldots P_{i_{2n+1}} : P_{i_j} \in E\},\ \ldots.$$

Then, clearly, $\bigcup_{i=0}^{\infty} A_i$ is a maximal doubly mutant set of the system, relative to all positive odd integers. To demonstrate that result recall that the set of all odd integers is a maximal mutant under addition and make use of the Unique Factorization Theorem of arithmetic.

Definition: Consider an algebraic system $(A, *_j)$ where $j \in J$ and J is non-empty. A set is said to be *p-tuply* mutant in $(A, *_j)$ where $0 < p \leq \text{card } J$, provided that the set is a mutant with respect to p and at most p of the composition laws.

Lemma: Let φ be a homomorphism from $(A, *_j)$ onto (B, o_k) for all $j \in J$ and $k \in K$ with, say, $\text{card } J \leq \text{card } K$. Let M be a maximal p-tuply mutant set of $(A, *_j)$ where $o < p \leq \text{card } J$ and $j \in J$. If $\varphi\overline{(M)} \subseteq \overline{\varphi(M)}$ then there exists a cardinal number q where $p \leq q \leq \text{card } K$ such that $\varphi(M)$ is a maximal q-tuply mutant set of (B, o_k) where $k \in K$.

Proof: Use the prelemma as an existence proof for multiply mutant sets. Then apply lemma 1.7 of (Mullin, 1961) *q* times.

SUMMARY

Two new and fundamental propositions, relating mutant sets to elementary number theory and general algebraic systems, are given.

LITERATURE CITED

MULLIN, A. A. 1960. *A Concept Concerning a Set with a Binary Composition Law*. Trans. Ill. St. Acad. Sci., 53 (3/4): 144-145.

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