

THE ELECTROMAGNETIC FIELD OF A BUNCHED ELECTRON BEAM

HERBERT H. SNYDER
Department of Mathematics
Southern Illinois University

ABSTRACT. — From qualitative considerations of phenomena taking place in the electron beams of traveling wave amplifiers (TWAs), general expressions for the current and charge densities in such beams are deduced, in the form of

superpositions of simple harmonic traveling waves. It is shown that the electromagnetic fields of such beams may be calculated from the classical retarded potentials, with an error which vanishes with increasing distance along the beam.

There is a class of microwave electron devices which leads one to interesting and useful problems in analysis. Among the most important of these is the traveling wave amplifier (TWA). Let us give a brief general description of such tubes. A TWA consists essentially of two ingredients, *viz.*, a wave guide structure along which electromagnetic waves can propagate, together with a stream of electrons flowing near to and along the whole length of the structure. The wave guide is designed in such a way that, among the class of waves which can propagate along it, there is at least one having a phase velocity small compared to the phase velocity of plane waves in vacuum. The electrons in the stream are given an average or drift velocity along the guide which is just equal to the phase velocity of the slow wave just mentioned. (A particle of the stream thus experiences a constant or at most a slowly varying phase of the field as it travels along the guide.) At the input end of the guide, matters are arranged so that a signal may be introduced, which then propagates along it. At the output end, the signal is extracted *via* a suitable matched load (so that there are only traveling waves, never standing waves, in the guide). It is found that when the beam and wave velocities are equal, or very nearly so, the amplitude of the extracted signal is large compared with the input, so that the device behaves as an amplifier. Indeed, it will amplify at all frequencies for which the mentioned phase and beam velocities are nearly equal, and its usefulness derives from the fact that it is possible to achieve such a correspondence in velocities over remarkably wide frequency intervals. For further technical details, we must refer to the works of Beck [1958], Slater [1950], Pierce [1950] and the literature cited there.

In general, analysis of phenomena in TWAs requires one to solve a non-linear system of hyperbolic partial differential equations. These are Maxwell's equations for the electro magnetic field vectors $\vec{E}$ and $\vec{H}$, supplemented by equations which relate $\vec{E}$ and $\vec{H}$ to the charge density δ and velocity $\vec{v}$ in the electron beam, together with suitable boundary conditions

at the edge of the beam and on the surface of the wave guide. Examples of such systems for particular TWAs may be found in the above cited works as well as in Chu and Jackson [1948]; Birdsall and Whinnery [1953].

The difficulty of obtaining even crude approximate solutions of the non-linear equations has led the writer to consider approximate *synthetic* methods of solution: instead of attempting to solve the non-linear system for the four unknown quantities $\bar{E}$, $\bar{H}$, δ , $\bar{v}$, we may instead consider the fields $\bar{E}$ and $\bar{H}$ which are associated with *given* beam densities and velocities δ and $\bar{v}$, wherein the latter contain certain parameters at least partially adjustable to given boundary conditions. The most important general phenomenon which takes place in the electron beam of a TWA (and, indeed, in certain other electron beam devices as well [3]) is *bunching* of the particles. Suppose — as is well-approximated in actual TWAs — that our beam of electrons is cylindrical in shape, occupying the domain $0 \leq r \leq B$, $z \geq 0$ in cylindrical coordinates r, ϕ, z (Fig. 1) (we may imagine $z = 0$ to be the location of the TWA input). Such a beam is said to be *bunched* if the particles are not distributed uniformly with distance along the z -axis, i.e., if the charge density $\delta = \delta(r, \phi, z, t)$, at any instant t , depends on z . (Such bunching occurs in TWAs because as the electrons travel along substantially in phase with the field, they tend to crowd into regions where the force on them is least—in the course of which, they give up energy to the field.)

In this paper, we study the fields $\bar{E}$ and $\bar{H}$ which are generated by cylindrical bunched beams. We first set forth some general relations from classical electromagnetic theory.

2. *Fields Of Given Charge And Current Distributions. Vector And Scalar Potentials.* We follow the well-established notation and terminology of Stratton [10], and use MKS units. (The dimensions of the physical quantities used are set forth explicitly.)

When the current density $\bar{J}$ (amperes/(meter)²) and the charge density δ (coulombs/(meter)³) are prescribed, the associated field vectors $\bar{E}$ (volts/meter) and $\bar{H}$ (ampere-turns/meter) are most simply obtained indirectly, *via* the vector and scalar potentials $\bar{A}$ (webers/meter) and Φ (volts). The latter satisfy the non-homogeneous wave equations

$$\nabla \nabla \cdot \bar{A} - \nabla \times \nabla \times \bar{A} - \mu \epsilon \frac{\partial^2 \bar{A}}{\partial t^2} = -\mu \bar{J} \quad (1)$$

$$\nabla^2 \Phi - \mu \epsilon \frac{\partial^2 \Phi}{\partial t^2} = -\frac{\delta}{\epsilon} \quad (2)$$

on any domain of vanishing conductivity, which is supposed. (ϵ = permittivity of the medium, farads/meter; μ = permeability of the medium,

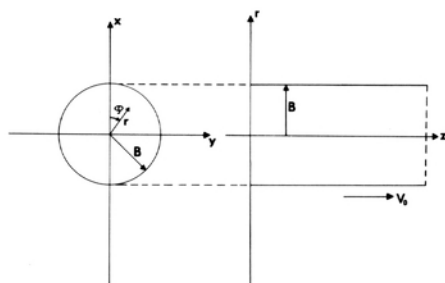


FIGURE 1.—Coordinates for cylindrical electron beams of radius $r = B$

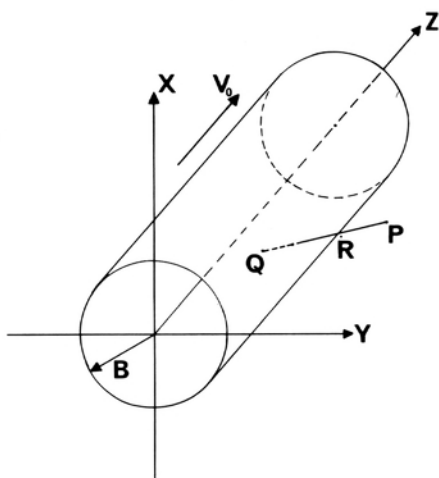
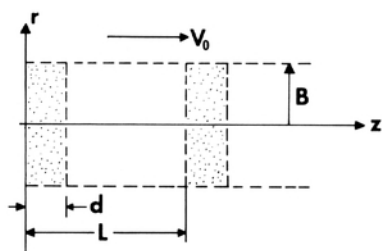


FIGURE 2.—Points P and Q, and distance R, in the retarded potentials (6), when D is the cylindrical domain of Fig. 1. Point Q runs over D while P represents an arbitrary but fixed point. (P may lie within as well as outside D.)

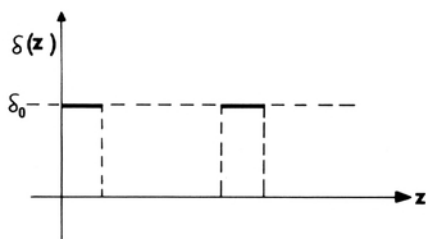


FIGURE 3.—A schematic diagram of events in a simple bunched beam. The charge is concentrated in periodically spaced discs of thickness d (Fig. 3A), with constant density in each disc (Fig. 3B)

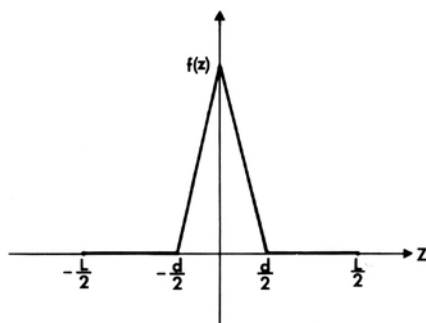


FIGURE 4.—Illustrating a simple periodic function $f(z)$ used to construct a charge density function.

henrys/meter, both being positive constants.) The vector operator on the left-hand side of (1) reduces to the Laplacian operating on the separate components of $\bar{A}$ only in rectangular coordinates. However, A_z is a cylindrical as well as a rectangular component, so that

$$\frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial A_z}{\partial r} + \frac{1}{r^2} \frac{\partial^2 A_z}{\partial \Phi^2} + \frac{\partial^2 A_z}{\partial z^2} - \mu \epsilon \frac{\partial^2 A_z}{\partial t^2} = -\mu J_z \quad (3)$$

If $\bar{A}$ and Φ are solutions of (1) and (2), then it may be shown [10] that the vectors

$$\bar{E} = -\nabla\Phi - \frac{\partial^2 \bar{A}}{\partial t^2}, \quad \mu\bar{H} = \nabla \times \bar{A} \quad (4)$$

are solutions of Maxwell's equations provided the densities $\bar{J}$ and δ satisfy the conservation law

$$\nabla \cdot \bar{J} + \frac{\partial \delta}{\partial t} = 0 \quad (5)$$

throughout their domain.

Well-known particular solutions of (1) and (2) are the *retarded potentials* of Kirchhoff and Lorentz ([5], [10]), described as follows. Let P be a fixed but arbitrary point, and Q any point of the domain D of $\bar{J}$ and δ . Then the potentials $\bar{A}$ and Φ at P , at any time t , are given by

$$\bar{A}(P, t) = -\frac{\mu}{4\pi} \int_D \frac{\bar{J}(Q, t^*)}{R} d\tau, \quad \Phi(P, t) = \frac{1}{4\pi\epsilon} \int_D \frac{\delta(Q, t^*)}{R} d\tau \quad (6)$$

where the integrals are taken over the entire domain D of the distribution (see Fig. 2, for D a cylindrical domain). In (6), R = distance from P to

Q , $t^* = t - \frac{R}{u}$, where $u = (\mu\epsilon)^{-1/2}$; u is the velocity of phase propagation

of plane waves in an unbounded medium with permeability μ and permittivity ϵ .

It is clear from (6) that in general $\bar{A}$ has as many non-vanishing components as does the current density $\bar{J}$. However, if $\bar{J} = \bar{i}_z J$ (in cylindrical coordinates), then — since $\bar{i}_z$ is a constant vector — $\bar{A}$ itself also has only an axial component $\bar{A} = \bar{i}_z A$, where

$$A(P, t) = \frac{\mu}{4\pi} \int_D \frac{J(Q, t^*)}{R} d\tau \quad (7)$$

In the sequel, we shall consider a class of densities for which the integral (6) for Φ and (7) for the z -component A of $\bar{A}$ are both explicitly evaluable. In other cases, of course, one must work directly with (1) and (2), taking advantage of whatever special features any particular problem may possess.

We remark, finally, that the beams of TWAs exist only in high vacua. The nature of the medium is accounted for in our equations by the constants ϵ and μ which, in vacuum, have the values $\epsilon_0 = 8.854 \cdot 10^{-12}$ farads/meter, $\mu_0 = 1.257 \cdot 10^{-6}$ henrys/meter. However, the beam may be

surrounded by media having different permittivities, in which case the *average* value of ϵ (say, over a portion of a plane transverse to the beam) may differ from its value for vacuum. It will therefore suffice for the present to make no special assumption about them other than that they are positive constants.

3. *Representations For Densities In Bunched Beams.* We suppose now that we have a beam of electrons confined to the cylindrical domain $D : 0 \leq r \leq B, z \geq 0$, as described above. Further, we suppose that the state of the beam is completely describable by a charge density δ and a one-dimensional current density $\bar{J} = \bar{i}_z J$, both of which are at least piece-wise differentiable functions of time t and position Q in D . If $\bar{v} = \bar{i}_z v$ is the velocity at any time and position in D , then $J = \delta v$. (It therefore suffices to specify any two of the three quantities J, δ and v .)

The simplest form of a bunched beam is found if we have a swarm of electrons, confined to D , all moving in the direction $z > 0$ with constant velocity v_0 , but in which the charge density at any particular time t depends on z . If $\delta = \delta(r, \varphi, z)$ at $t = 0$ then, at any other time we have $\delta = \delta(r, \varphi, z - v_0 t)$, on account of the constant velocity v_0 at every point of the stream. Then the current density J at any point is simply δv_0 . In turn, the simplest useful variation of δ for our purposes is that of a constant component on which a *periodic* function of $z - v_0 t$ is superposed. For example, the beam might consist of discs of charge of thickness d , at constant distance apart, with the whole configuration moving with constant velocity $v_0 > 0$. Then, at $t = 0$, the beam has the appearance sketched in Fig. 3, in which we have supposed the density to be constant ($= \delta_0$) in each disc. In this case,

$$\begin{aligned} \delta(z) &= \delta_0, 0 \leq z \leq d, \\ &= 0, d < z < L \end{aligned} \quad (8)$$

and if we define $\delta(z + L) = \delta(z)$, then $\delta(z)$ has a Fourier development

$$\delta(z) = \sum_{m=-\infty}^{\infty} A_m e^{i \frac{2m\pi}{L} z}$$

and then

$$\delta(z, t) = \sum_{m=-\infty}^{\infty} A_m e^{i \frac{2m\pi}{L} (z - v_0 t)} \quad (9)$$

where

$$A_m = \frac{1}{L} \int_0^L \delta(z) e^{-i(2m\pi/L)z} dz, m = 0, \pm 1, \pm 2, \dots$$

Let us notice an important restriction: if our expressions for δ are to represent *electron* densities, then we must have $-\delta \geq 0$, according to the usual sign convention for the charge of an electron. With this point in mind, we may assign to δ any other periodic distribution having a Fourier representation instead of the simple example (8).

Now expression (9) is a superposition of simple harmonic traveling

waves of the form $\exp i(\beta_m z - \omega_m t)$, with $\beta_m = \frac{2m\pi}{L}$, $\omega_m =$

$v_0 \beta_m$, so that the component wave of index $m=0$ has phase-shift β_m radians/meter and oscillates at the frequency $f_m = \omega_m/2\pi$ cycles/second. In particular, the fundamental frequency $\omega_1 = 2\pi v_0/L$, increasing directly with the beam velocity v_0 and inversely with the period L .

More generally, we need not assume that the density is strictly periodic in z , but may instead assign to each component of frequency ω_m its own particular phase β_m , and write

$$\delta = \sum_{-\infty}^{\infty} A_m e^{i(\beta_m z - \omega_m t)} \quad (10)$$

and, in fact, β_m is generally a function of the fundamental frequency ω_1 . Furthermore, there is no reason why the amplitudes A_m need be constant across the beam and, in fact, they are in general functions of position r, ϕ transverse to the direction of propagation. Thus we write the more general expression, in place of (10),

$$\delta = \delta(r, \phi, z, t) = \sum A_m(r, \phi) e^{i(\beta_m z - \omega_m t)} \quad (11)$$

Further, in order eventually to build up suitable models of phenomena in TWAs, we need densities like (11) but in which the amplitude of each wave varies exponentially with z , say in the form

$$e^{\alpha_m(z)} e^{i(\beta_m z - \omega_m t)} \quad (12)$$

where the functions $\alpha_m(z)$ are continuous. Unfortunately, in (12) we have lost the representation of δ by simple harmonic traveling waves. If, however, we are only interested in events taking place over some finite length of the beam, we may suppose that $\alpha_m(z)$ (and hence $\exp \alpha_m(z)$) is periodic with a period D comparable to the length in question (i.e., with a period D large compared with the period L above, or compared with the largest

of the periods $2\pi/\beta_m$). Then if $\exp \alpha_m(z)$ has a Fourier development,

$$\exp \alpha_m(z) = \sum_{k=-\infty}^{\infty} B_{mk} e^{i \frac{2\pi k}{D} z} \quad (13)$$

we substitute (13) into (11) and (12) and obtain a double series

$$\sum_{m,k} A_m(r, \phi) B_{mk} e^{i(\beta_{mk} z - \omega_m t)} \quad (14)$$

where $\beta_{mk} = \beta_m + \frac{2\pi k}{D}$. Our representation of δ by simple harmonic

traveling waves is thus formally restored, while permitting us to assign quite general values to the factors $\alpha_m(z)$ over any given length D .

Finally, it may be the case that when the charge density — considered as a function of t — is resolved into its Fourier components, a *single* traveling wave at each frequency ω_m is inadequate to describe the distribution. Thus, at any harmonic $\omega_m = m\omega_1$ of the fundamental frequency ω_1 , one may have several traveling waves. Then (14) is replaced by the more general expression

$$\begin{aligned} \sum_{m,k,p} A_{mkp}(r, \phi) e^{i(\beta_{mkp} z - \omega_m t)} &= \sum_{m,k} A_{mk0} e^{i(\beta_{mk0} z - \omega_m t)} \\ &+ A_{mk1} e^{i(\beta_{mk1} z - \omega_m t)} + A_{mk,-1} e^{i(\beta_{mk,-1} z - \omega_m t)} + \dots \end{aligned} \quad (15)$$

It only remains to notice that since the beam occupies the whole cylinder $0 \leq r \leq B$, any physically realizable density must be periodic in ϕ with period 2π . Expanding the factor $A_{mkp}(r, \phi)$ in (15), we have a representation of the form

$$\delta(r, \phi, z, t) = \sum A_{mkpn}(r) e^{in\varphi} e^{i(\beta_{mkpn} z - \omega_m t)} \quad (16)$$

where the sum is taken over all the indices m, k, p, n . If we assume a similar expression for the velocity v ,

$$v(r, \varphi, z, t) = \sum B_{mkpn}(r) e^{in\varphi} e^{i(\beta_{mkpn} z - \omega_m t)} \quad (17)$$

then the current density $J = \delta v$ is formally of the same type with amplitude factors $C_{mkpn}(r)$ given by suitable sums of products of the A and B coefficients in (16) and (17).

Let us see how these representations correspond to what is known from existing theories of the TWA. These are all linearized, low-level theories (e.g., [4], [8]) from which the following general picture of events emerges: at a fixed frequency ω , the axial component E_z of the electric field, in the *absence* of the electron beam, is given by an expression of the type

$$E_z = \sum_{n,m} E_n(r) e^{\frac{i n \varphi}{e}} e^{i(\beta_{nm} z - \omega t)} \quad (18)$$

Then when a cylindrical beam with only z -components of density and velocity is injected into the tube, each component wave of (18) breaks up into at least *four* waves, with the same number of waves in the representation for the charge density and velocity induced by the interaction between beam and wave. Moreover, the amplitude of each of these waves depends on z , either growing or decaying exponentially like $e^{\pm \alpha^* z}$ ($\alpha = \text{const.}$), or remaining constant, as the whole field travels in the direction $z > 0$. It will be seen that in our heuristic discussion leading up to (16) and (17), these are precisely the possibilities which have been accounted for and, in fact, are even of a more general class than the density and velocity expressions produced by linear analyses. (For example, we have allowed for the fact that the amplitude change with z may be of a much more general character than merely increasing or decreasing like $e^{\pm \alpha^* z}$.)

Now consider the space $\mathcal{F}$ of all pairs of well-defined expressions of the form

$$\begin{aligned} \delta(r, \varphi, z, t) &= \sum A(r) e^{\frac{i n \varphi}{e}} e^{i(\beta z - \omega t)} \\ J(r, \varphi, z, t) &= \sum B(r) e^{\frac{i n \varphi}{e}} e^{i(\beta z - \omega t)} \end{aligned} \quad (19)$$

with sums in (19) taken over suitable indices, and defined on $0 \leq r \leq B$, and where each pair satisfies the conservation law

$$\frac{\partial J}{\partial z} + \frac{\partial \delta}{\partial t} = 0 \quad (20)$$

By a bunched beam, we shall understand any pair δ, J of $\mathcal{F}$.

4. *The Fields of Bunched Beams.* From (6) and (7), we have the integrals

$$A(P, t) = -\frac{\mu}{4\pi} \int_D \frac{J(Q, t^*)}{R} d\tau, \quad \Phi(P, t) = \frac{1}{4\pi\epsilon} \int_D \frac{\delta(Q, t^*)}{R} d\tau \quad (21)$$

for the potentials, from which the fields may be derived. Consider, for example, the scalar potential $\Phi(P, t)$. The density δ of (19) evaluated at

t^* is

$$\delta(r, \varphi, z, t^*) = \sum A(r) e^{\frac{i n \varphi}{e}} e^{\frac{i(\beta z - \omega t)}{e}} e^{\frac{i k R}{e}}$$

where $k = \omega/u$. If we substitute this expression in the integral (21), we have, upon interchanging orders of summation and integration

$$4\pi\epsilon \Phi(P, t) = \int_D \frac{e^{i k R}}{R} [A(r) e^{\frac{i n \varphi}{e}} e^{\frac{i(\beta z - \omega t)}{e}}] d\tau \quad (22)$$

Expression (22) shows that the integrals (21) define linear operators on the densities δ and J . It suffices, therefore, to consider integrals of the form

$$I(P, t) = \int_D \frac{e^{i k R}}{R} \delta(Q, t) d\tau \quad (23)$$

where $\delta(Q, t)$ now represents an arbitrary one of the elementary components of the densities (19), and from which we then construct the potentials of "arbitrary" distributions in $\mathcal{F}$ by superposition.

To evaluate (23), let ρ, θ, ξ be cylindrical coordinates of Q in D : $0 \leq \rho \leq B$, $z \geq 0$, and r, φ, z coordinates of the arbitrary but fixed point P (which may or may not be in D). Then we have

$$\delta(Q, t) = A(\rho) e^{\frac{i n \theta}{e}} e^{\frac{i(\beta \xi - \omega t)}{e}}$$

where n is a fixed but arbitrary integer; since

$$R^2 = r^2 + \rho^2 - 2r\rho \cos(\theta - \varphi) + (\xi - z)^2$$

(23) becomes

$$I = e^{-i\omega t} \int_0^B A(\rho) \rho d\rho \int_0^{2\pi} e^{\frac{i n \theta}{e}} d\theta \int_0^\infty \frac{e^{i k R}}{R} e^{\frac{i \beta \xi}{e}} d\xi \quad (24)$$

First, consider the integral with respect to ξ in (24),

$$\int_0^\infty \frac{e^{i k R}}{R} e^{\frac{i \beta \xi}{e}} d\xi \quad (24a)$$

and make the change of variable $\zeta = x + z$, giving

$$\int_0^{\infty} \dots = e^{i\beta z} \int_{-z}^{\infty} \frac{e^{ikR}}{R} e^{i\beta x} dx, \quad R^2 = r^2 + \rho^2 - 2r\rho \cos(\theta - \Phi) + x^2 \quad (25)$$

To evaluate (5), we use Sommerfeld's integral representation [6] for

$\frac{e^{ikR}}{R}$, viz.,

$$\frac{e^{ikR}}{R} = \int_0^{\infty} J_0(\tau \sqrt{r^2 + \rho^2 - 2r\rho \cos(\theta - \Phi)}) \frac{e^{-|x| \sqrt{\tau^2 - k^2}}}{\sqrt{\tau^2 - k^2}} \tau d\tau \quad (26)$$

Upon substituting in (25) and interchanging orders of integration, (25) becomes

$$e^{i\beta z} \int_0^{\infty} J_0(\tau \sqrt{\dots}) \frac{\tau d\tau}{\sqrt{\tau^2 - k^2}} \int_{-z}^{\infty} \frac{e^{i\beta x}}{e^{-|x| \sqrt{\tau^2 - k^2}}} dx \quad (27)$$

For the integral with respect to x , we write

$$\int_{-z}^{\infty} = \int_{-z}^{\infty} + \int_{-\infty}^{-z} - \int_{-\infty}^{-z} = \int_{-\infty}^{\infty} - \int_{-\infty}^{-z}$$

so that

$$\int_{-z}^{\infty} \frac{e^{i\beta x}}{e^{-|x| \sqrt{\tau^2 - k^2}}} dx = \frac{2\sqrt{\tau^2 - k^2}}{\tau^2 + \beta^2 - k^2} + e^{-i\beta z} \frac{e^{-z \sqrt{\tau^2 - k^2}}}{i\beta + \sqrt{\tau^2 - k^2}}$$

and we have for (25)

$$\int_0^{\infty} e^{\frac{ikR}{R}} e^{i\beta\zeta} d\zeta = e^{i\beta z} \int_0^{\infty} J_0(\tau\sqrt{\dots}) \frac{2\tau d\tau}{\tau^2 + \beta^2 - k^2} \quad (28)$$

$$+ \int_0^{\infty} J_0(\tau\sqrt{\dots}) \frac{e^{-z\sqrt{\tau^2 - k^2}}}{(i\beta + \sqrt{\tau^2 - k^2})\sqrt{\tau^2 - k^2}} \tau d\tau \quad (28)$$

For the first integral on the right of (28), we have the addition theorem for the Bessel function $J_0(x)$ [11]

$$J_0(\tau\sqrt{\dots}) = \sum_{m=-\infty}^{\infty} J_m(\tau r) J_m(\tau \rho) e^{im(\varphi - \theta)}$$

an expansion which is uniformly and absolutely convergent for all real values of the arguments. Hence

$$\int_0^{\infty} J_0(\tau\sqrt{\dots}) \frac{2\tau d\tau}{\tau^2 + \beta^2 - k^2} = 2 \sum_m e^{im(\varphi - \theta)} \int_0^{\infty} \frac{J_m(\tau r) J_m(\tau \rho) \tau d\tau}{\tau^2 + \beta^2 - k^2} \quad (28a)$$

Let us define

$$\begin{aligned} \gamma^2 &= \beta^2 - k^2, \beta > k, \\ &= k^2 - \beta^2, \beta < k. \end{aligned}$$

Then the integral in (28a) has the value [11, p. 429]

$$\left. \begin{aligned} &I_m(\gamma r) K_m(\gamma \rho), \rho > r \\ &I_m(\gamma \rho) K_m(\gamma r), \rho < r \end{aligned} \right\} \beta > k$$

$$\left. \begin{aligned} &\frac{\pi i}{2} J_m^{(1)}(\gamma r) H_m^{(1)}(\gamma \rho), \rho > r \\ &\frac{\pi i}{2} J_m^{(1)}(\gamma \rho) H_m^{(1)}(\gamma r), \rho < r \end{aligned} \right\} \beta < k \quad (29)$$

where $J_m(x)$, $H_m(x)$, $I_m(x)$, $K_m(x)$ are the ordinary Bessel function, Hankel function, modified Bessel function, and modified Hankel function, respectively, all of order m . Let $F_m(\gamma r, \gamma \rho)$ denote the value of the integral. Then we have for (28)

$$\int_0^{\infty} \frac{e^{ikR}}{R} e^{i\beta\zeta} d\zeta = 2e^{i\beta z} \sum_m^m e^{im(\phi-\theta)} F_m(\gamma r, \gamma \rho) \quad (30)$$

$$+ \int_0^{\infty} J_0(\tau \sqrt{\dots}) \frac{e^{-z\sqrt{\tau^2-k^2}} \tau d\tau}{(i\beta + \sqrt{\tau^2-k^2}) \sqrt{\tau^2-k^2}}$$

Let us notice, in (30), that the integral on the right-hand side comes from the fact that the beam "begins" at $z = 0$. If the domain of the densities had been taken to be the unbounded cylindrical domain $0 \leq r \leq B$, $-\infty < z < \infty$, then (24a) would have been replaced by

$$\int_{-\infty}^{\infty} \frac{e^{ikR}}{R} e^{i\beta\zeta} d\zeta$$

and consequently the integral in (27) with respect to x would also have extended over $-\infty < x < \infty$, leaving us with just the series part of the right-hand side of (30). The mentioned integral thus gives that part of the field arising from the fact that our beam model is discontinuous at $z = 0$: for $z < 0$, the medium is free of charge, but for $z > 0$ it is not. We show in the Appendix that this part of the integral is vanishingly small except near $z = 0$, and tends to zero as $z \rightarrow \infty$, uniformly over the beam cross-section. Thus for sufficiently large values of z (30) is, approximately,

$$\int_0^{\infty} \frac{e^{ikR}}{R} e^{i\beta\zeta} d\zeta = 2e^{i\beta z} \sum_{m=-\infty}^{\infty} e^{im(\phi-\theta)} F_m(\gamma r, \gamma \rho) \quad (31)$$

Then (24) becomes

$$I(r, \varphi, z, t) = 2e^{i(\beta z - \omega t)} \sum_{m=-\infty}^{\infty} e^{im\varphi} \int_0^{2\pi} e^{i(n-m)\theta} d\theta \int_0^B A(\rho) F_m(\gamma r, \gamma \rho) \rho d\rho$$

every term of which vanishes identically except when $m = n$:

$$I(r, \varphi, z, t) = 4\pi e^{i(\beta z - \omega t)} e^{in\varphi} \int_0^B A(\rho) F_n(\gamma r, \gamma \rho) \rho d\rho \quad (32)$$

We therefore have the following result: let $D: 0 \leq r \leq B, z > 0$, and

let $\delta(Q, t) = A(r) e^{in\varphi} e^{i(\beta z - \omega t)}$ on D . Then, if R is the distance from $Q (= \rho, \theta, \zeta)$ in D to an arbitrary point $P (= r, \varphi, z)$,

$$\int_D \frac{e^{ikR}}{R} \delta(Q, t) d\tau = 4\pi e^{in\varphi} e^{i(\beta z - \omega t)} \int_0^B A(\rho) F_n(\gamma r, \gamma \rho) \rho d\rho \quad (33)$$

where $F_n(\gamma r, \gamma \rho)$ is given by expressions (29).

To evaluate the integral in (33) (denote it by $R_n(\gamma r)$), we have two cases, according as β is $>k$ or $<k$. (It is easy to show that the expressions (29a,b) are continuous at $\beta = k$.) What is the physical significance of these two possibilities? We have already remarked that $u = (\mu\epsilon)^{-1/2}$ is the phase velocity of plane waves in an empty unbounded medium of constants μ, ϵ , i.e., a characteristic velocity for the medium itself. The phase velocity v of the elementary wave with phase factor $\exp i(\beta z - \omega t)$ is just ω/β . We thus have

$$\frac{v}{u} < 1, \beta > k, \\ > 1, \beta < k$$

so that in the first case the phase of the elementary beam wave is traveling in the direction $z > 0$ with velocity less than that for plane waves in the

medium, and in the second case, *greater*. Let us consider the first case, which is relevant to TWAs. There are again two possibilities, according as P lies within ($r < B$) or without ($r > B$) the beam. In the former case,

$$R_n(\gamma r) = K_n(\gamma r) \int_0^r A(\rho) I_n(\gamma \rho) \rho d\rho + I_n(\gamma r) \int_r^B A(\rho) K_n(\gamma \rho) \rho d\rho \quad (34)$$

and in the latter

$$R_n(\gamma r) = K_n(\gamma r) \int_0^B A(\rho) I_n(\gamma \rho) \rho d\rho \quad (35)$$

Comparing the expressions (21), (23) and (33), the scalar potential $\Phi(P, t)$ of the charge density wave is given by

$$\Phi(P, t) = \frac{1}{\epsilon} R_n(\gamma r) e^{\frac{i n \varphi}{e}} e^{i(\beta z - \omega t)} \quad (36a)$$

and the axial component $A(P, t)$ of the vector potential of the current

density wave $J = B(r) e^{\frac{i n \varphi}{e}} e^{i(\beta z - \omega t)}$ is, similarly,

$$A(P, t) = \mu S_n(\gamma r) e^{\frac{i n \varphi}{e}} e^{i(\beta z - \omega t)} \quad (36b)$$

where in (36)

$$\left. \begin{matrix} R_n(\gamma r) \\ S_n(\gamma r) \end{matrix} \right\} = \int_0^B \left. \begin{matrix} A(\rho) \\ B(\rho) \end{matrix} \right\} F_n(\gamma r, \gamma \rho) \rho d\rho \quad (37)$$

Let us now calculate the fields $\overline{E}$ and $\overline{H}$ produced by the densities. The formulas (4) reduce to

$$\begin{aligned} E_r &= -\frac{\partial \Phi}{\partial r}, E_\varphi = -\frac{1}{r} \frac{\partial \Phi}{\partial \varphi}, E_z = -\frac{\partial \Phi}{\partial z} - \frac{\partial A}{\partial t} \\ \mu H_r &= \frac{1}{r} \frac{\partial A}{\partial \varphi}, \mu H_\varphi = -\frac{\partial A}{\partial r}, H_z = 0 \end{aligned}$$

Applying these to (36), we obtain in cylindrical coordinates

$$\left. \begin{aligned} \mathbf{E} &= -\frac{\gamma}{r} \frac{R'}{\epsilon_n} (\gamma r) \\ \mathbf{E} &= -\frac{\text{in}}{\varphi} \frac{R}{r n} (\gamma r) \\ \mathbf{E} &= -\frac{i\beta}{z} \frac{R}{\epsilon_n} (\gamma r) + i\omega\mu \frac{S}{n} (\gamma r) \\ \mathbf{H} &= \text{in} \frac{S}{r n} (\gamma r) \\ \mathbf{H} &= -\gamma \frac{S'}{\varphi n} (\gamma r) \\ \mathbf{H} &= 0 \end{aligned} \right\} \times e^{\text{in}\varphi} e^{i(\beta z - \omega t)} \quad (38)$$

(in which the prime means derivative with respect to γr). It follows that the field vectors $\overline{\mathbf{E}}$ and $\overline{\mathbf{H}}$ associated with densities given by the sum of an arbitrary finite number of elementary densities, is just the corresponding sum of expressions (38). Let us remark also that potentials (36) are particular solutions of the non-homogeneous equations (2) and (3) for the elementary densities.

We conclude with an example. Let $f(z)$ be the even function of period L defined by (Fig. 4)

$$\left. \begin{aligned} f(z) &= 1 - \frac{2z}{d}, \quad 0 < z < \frac{d}{2}, \\ &= 0, \quad \frac{d}{2} < z < \frac{L}{2}, \end{aligned} \right\} \quad 0 < d < L \quad (39)$$

$$f(-z) = f(z), \quad f(z + L) = f(z)$$

Computation gives

$$f(z) = \frac{d}{2L} + \frac{d}{L} \sum_{m=1}^{\infty} \left(\frac{\sin \theta_m}{\theta_m} \right)^2 \cos \frac{2\pi m}{L} z$$

$$= \frac{d}{2L} \sum_{m=-\infty}^{\infty} \left(\frac{\sin \theta_m}{\theta_m} \right)^2 e^{i \frac{2m\pi}{L} z}, \quad \theta_m = \frac{m\pi d}{2L}$$

Then the expression

$$\frac{d}{2L} \sum_{m=-\infty}^{\infty} \left(\frac{\sin \theta_m}{\theta_m} \right)^2 e^{i \frac{2m\pi}{L} (z - v_0 t)}$$

represents a wave traveling in the direction $z > 0$ with constant velocity v_0 ($< u$) and which coincides with the function (39) at $t = 0$. Let us further modify the expression by introducing a dependence on the radius, say

$$1 - \left| \frac{r}{B} \right|^{|m|}$$

for the m -th harmonic. Then

$$\delta(r, \varphi, z, t) = -\frac{\delta_0 d}{2L} \sum_{m=-\infty}^{\infty} \left(\frac{\sin \theta_m}{\theta_m} \right)^2 \left(1 - \left| \frac{r}{B} \right|^{|m|} \right) e^{i \frac{2m\pi}{L} (z - v_0 t)} \quad (40)$$

($0 \leq r \leq B$) where $\delta_0 = \text{const.} > 0$, is a density of the sort considered above, and $J = \delta v_0$ the corresponding current density.

Comparing (40) with (33) and subsequent expressions, we set

$$n = 0, \quad \beta = \frac{2m\pi}{L}, \quad \omega_m = v_0 \beta, \quad A(r) \sim 1 - \left| \frac{r}{B} \right|^{|m|}$$

Let us compute, for instance, E_z at points outside the beam. We have

$$\begin{aligned} R_0(\gamma r) &= K_0(\gamma r) \int_0^B \left(1 - \left(\frac{\rho}{B} \right)^{|m|} \right) I_0(\gamma \rho) \rho d\rho \\ &= F_0(\gamma B) K_0(\gamma r) \end{aligned}$$

Then (38) gives us

$$E_z = \frac{\delta_0 d}{2L} \sum_{m=-\infty}^{\infty} \left(i\omega\mu v_0 - \frac{i\beta_m}{\epsilon} \right)$$

$$\left(\frac{\sin\theta_m}{\theta_m} \right)^2 F_O (\gamma_m B) K_O (\gamma_m r) e^{i \frac{2m\pi}{L} (z-v_0 t)}$$

(where $\gamma_m^2 = \beta_m^2 - k_m^2 = \left(\frac{2m\pi}{L} \right)^2 \left(1 - \left(\frac{v_0}{u} \right)^2 \right)$), in which we have neglected the static part ($m = 0$) of the field.

In a subsequent paper, we shall take up the question of how the densities may be chosen in order to represent phenomena in TWAs.

APPENDIX

We consider the second term on the right-hand side of (30), *viz.*, the improper integral

$$T = \int_0^{\infty} J_0(M\tau) \frac{e^{-z\sqrt{\tau^2-k^2}} \tau d\tau}{(i\beta + \sqrt{\tau^2-k^2}) \sqrt{\tau^2-k^2}}$$

where $M = \sqrt{\tau^2 + \rho^2 - 2\rho\epsilon\cos(\theta-\varphi)}$, in which β and k are fixed non-zero real numbers $\beta > k$. Near $\tau = k$, the integrand is proportional to $(\tau - k)^{-1}$, which is integrable. In fact we can write

$$T = \int_0^k + \int_k^{\infty}$$

In the first integral, set $\tau = k \sin x$, in the second, $\tau = k \cosh x$. Then

$$T = - \int_0^{\pi/2} J_0(Mk \sin x) e^{-izk \sin x} \frac{k \sin x \, dx}{\beta + k \cos x}$$

$$+ \int_0^{\infty} J_0(Mk \cosh x) e^{-zk \sinh x} \frac{k \cosh x \, dx}{i\beta + k \sinh x}$$

$$= T_1 + T_2$$

and since $\beta > k$, the first integrand is continuous. Consider the first integral T_1 ; we have the identity

$$e^{-izk \sin x} = \sum_{n=-\infty}^{\infty} i^{-n} J_n(kz) e^{inx}$$

the right-hand side of which is uniformly and absolutely convergent for every real z . Then

$$-T_1 = \sum_{n=-\infty}^{\infty} i^{-n} J_n(kz) \int_0^{\pi/2} J_0(Mk \sin x) e^{inx} \frac{k \sin x \, dx}{\beta + k \cos x}$$

and

$$|T_1| \leq k \sum_{n=-\infty}^{\infty} |J_n(kz)| \int_0^{\pi/2} |J_0(Mk \sin x)| \frac{|\sin x| \, dx}{|\beta + k \cos x|}$$

Since $|J_0(Mk \sin x)| \leq 1$, we have

$$|T_1| \leq \frac{\pi}{2} \frac{k}{\beta - k} \sum_{n=-\infty}^{\infty} |J_n(kz)|$$

But every term of the last series tends to zero as $z \rightarrow \infty$.

For the second integral T_2 , we have

$$|T_2| \leq \int_0^{\infty} e^{-zk \sinh x} \frac{k \cosh x \, dx}{\sqrt{\beta^2 + k^2 \sinh^2 x}}$$

If we set $u = k \sinh x$, the last integral becomes

$$\int_0^{\infty} \frac{e^{-uz} du}{\sqrt{\beta^2 + u^2}} = J_0(\beta z)$$

which also vanishes as $z \rightarrow \infty$.

LITERATURE CITED

1. BECK, A. H. W. 1958. *Space-Charge Waves And Slow Electromagnetic Waves*, New York, (Pergamon Press)
2. BIRDSALL, C. K. and J. R. WHINNERY 1953. "Waves in an electron stream with general admittance walls", *Journal of Applied Physics*, vol. 24, pp. 314-323.
3. BRONWELL, A. B., and R. E. BEAM 1947. *Theory And Application Of Microwaves*, New York, (McGraw-Hill).
4. CHU, L. J., and J. D. JACKSON 1948. "Field theory of traveling-wave tubes", *Proceedings Of The Institute Of Radio Engineers*, vol. 36, pp. 230-233.
5. LORENTZ, H. A. 1952. *The Theory Of Electrons*, New York, (Dover reprint ed.)
6. MAGNUS, W., and F. OBERHETTINGER 1949. *Formulas And Theorems For The Special Functions Of Mathematical Physics*, New York, (Chelsea reprint ed.).
7. PIERCE, J. R. 1950. *Traveling-Wave Tubes*, New York, (Van Nostrand).
8. RYDBECK, O. E. H. 1948. "The theory of the traveling-wave tube", *Ericsson Technics*, No. 46, Telefonaktiebolaget L. M. Ericsson, Stockholm.
9. SLATER, J. C. 1950. *Microwave Electronics*, New York, (Van Nostrand).
10. STRATTON, J. A. 1941. *Electromagnetic Theory*, New York, (McGraw-Hill).
11. WATSON, G. N. 1958. *A Treatise On The Theory Of Bessel Functions*, Cambridge, (Cambridge U. Press, 2nd ed.).

Manuscript received July 6, 1967.