VARIOUS METHODS OF SOLVING A FIRST ORDER PERTURBATION EQUATION

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ABSTRACT.—A general procedure is described for obtaining the complementary function of a linear, second-order, inhomogeneous differential equation. This method is applicable to a majority of the radial equations of quantum theory. Three methods of finding the particular integral are discussed. The techniques are unified by the use of an example from perturbation theory.

The purpose of this paper is to present and compare several methods of solving a differential equation which occurs in quantum mechanical problems. None of the techniques are new but this paper presents them in such a manner that a beginner can choose the best method for his problem. Further, no completely worked out example is available in the literature so that the present paper provides a guide post for the beginning research worker. The example which is used occurred in a perturbation treatment of the refractive index of helium (Musulin and Epstein, 1964).

PRELIMINARIES

Since the notation of that report is unduly cumbersome for illustrating the points in the solution of a single differential equation, the following changes are made (the new notation appearing on the right).

$$H_{00} = H_0 \qquad g_1^{(0)} = g$$

$$E_0^0 = E_0 \qquad g_2^{(0)} = f$$

$$\Psi_{0+0} = \Psi^0$$

$$\Psi_{1+0}^{(0)} = \Psi^1$$

$$G^+ = V$$

Throughout this report Hartree atomic units are used (1 bohr = 0.52917 Å and 1 hartree = 27.2097 e.v. (Manne, 1963)).

The Basic Equations.—The basic equations resulting from a triple perturbation in time, $^1/\zeta$, and ω (the angular frequency) are

$$(H_0 - E_0)\ \Psi^0 = 0 \qquad (1)$$

$$(H_0 - E_0)\ \Psi^1 = -V\Psi^0$$

where

$$H_0 = h_1 + h_2$$

$$= [-\tfrac{1}{2}\nabla_1^2 - \zeta/r_1] + [-\tfrac{1}{2}\nabla_2^2 - \zeta/r_2]$$

$$E_0 = -\zeta^2/n^2$$

$$\Psi^0 = \Psi_1\ \Psi_2$$

$$= \left[\left(\frac{\zeta^{3/2}}{\sqrt{\pi}}\right) e^{-\zeta r_1}\right]\left[\left(\frac{\zeta^{3/2}}{\sqrt{\pi}}\right) e^{-\zeta r_2}\right]$$

$$V = V_1 + V_2$$

$$= [-r_1 P_1(\cos\theta_1)] + [-r_2 P_1(\cos\theta_2)]$$

The electric field, $\mathcal{E}$, which multiplies V_1 and V_2 is omitted since it may be considered as a perturbation parameter.

Partial Legendre Expansion.—The solution of Equation (1) is now discussed in detail. This problem is reduced to a problem of obtaining appropriate radial functions after Ψ^1 is partially expanded in Legendre Polynomials (Hirschfelder, *et. al.*, 1963, pp. 30-31). In this case

$$\Psi^1 = [g(r_1)P_1(\cos\theta_1)\,P_0(\cos\theta_2) + g(r_2)P_0(\cos\theta_1)\,P_1(\cos\theta_2) + f(r_1)P_0(\cos\theta_1)\,P_0(\cos\theta_2) + f(r_2)P_0(\cos\theta_1)\,P_0(\cos\theta_2)]\,\Psi^0 \quad (2)$$

Substitution of Equation (2) into Equation (1) leads, after the appropriate integrations, to

$$-\frac{1}{2}g''(r_i) - \frac{1}{2}\left(\frac{2}{r_i} - 2\zeta\right)g'(r_i) + \frac{1}{r_i^2}g(r_i) = r_i \quad (3)$$

$$-\frac{1}{2}f''(r_i) - \frac{1}{2}\left(\frac{2}{r_i} - 2\zeta\right)f'(r_i) = K \quad (4)$$

where i takes the values 1 and 2. The remainder of the report illustrates the different approaches to solving Equation (3).

It should be pointed out that the same techniques which solve Equation (3) prove that the only acceptable solution for Equation (4) is

$$f(r_i) = 0$$

However, it should be emphasized that in many instances it is not necessary to obtain a formal solution for every unknown radial function. A particular problem should be first analyzed to see if each term of the partial Legendre expansion contributes to the matrix elements to be evaluated.

Confluent Hypergeometric Equation

Recognizing the Equation.—Equation (3) is characterized as a linear, second-order differential equation. It is also called an inhomogeneous differential equation because the term appearing on the right hand side contains neither the unknown function g or its derivatives. It can also be remarked that Equation (3) is not an eigenvalue equation.

The solution of an inhomogeneous differential equation relies heavily upon the solution of the associated homogeneous equation; that is, the equation whose right hand side is zero. For the sake of clarifying notation, the unknown function in the associated homogeneous equation is designated by G(r).

The homogeneous equation associated with Equation (3), after multiplication by −2, is (dropping the variable subscripts for convenience)

$$G''(r) + \left(\frac{2}{r} - 2\zeta\right)G'(r) - \frac{2}{r^2}G(r) = 0 \quad (5)$$

The general second-order linear differential equation can be written

$$y'' + P(x)y' + Q(x)y = X(x) \quad (6)$$

The substitution (Burington and Torrance, 1939),

$$y = e^{-\frac{1}{2}\int P\,dx}\,w \quad (7)$$

changes Equation (6) into a second order linear differential equation which does not contain terms in the first derivative. Although the general solution of Equation (3) is not readily recognized, often, the equation, which results from a transformation of the type given in Equation (7), for radial problems is a standard differential equation. In particular, it is the differential equation whose solution is the confluent hypergeometric function (4).

For equation (3) the substitution

$$G = \frac{1}{r}e^{\zeta r}w \quad (8)$$

carries equation (5) into

$$w'' + \left[-\zeta^2 + \frac{2\zeta}{r} - \frac{2}{r^2}\right]w = 0 \quad (9)$$

which is a hybrid standard form. The standard equation is usually written

$$w'' + \left[-\mu^2 + (\gamma - 2\alpha)\frac{\mu}{r} + \frac{\gamma(2-\gamma)}{4r^2}\right]w = 0 \quad (10)$$

Another transformation results from the change of variable in Equation (9)

$$z = 2\zeta r$$

which gives

$$w'' + \left[-\frac{1}{4} + \frac{1}{z} - \frac{2}{z^2} \right] w = 0 \quad (11)$$

The corresponding standard form, Whittaker's equation, (Jeffreys and Jeffreys, 1950) is

$$w'' + \left[-\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2} \right] w = 0 \quad (12)$$

Finally, a third transformation results from substituting in Equation (11)

$$w = e^{-\frac{1}{2}z} z^2 u \quad (13)$$

which yields

$$zu'' + (4 - z)u' - u = 0 \quad (14)$$

and this corresponds to the standard form, Kummer's equation, (Jeffreys and Jeffreys, 1950)

$$xu'' + (\gamma - x)u' - \alpha u = 0 \quad (15)$$

Although it is often more convenient to use the solutions to Kummer's equation, it is usually more direct to proceed to Equation (10) which is a variant of Whittaker's equation. Consequently, the solutions to both Equations (12) and (15) are now discussed.

The Kummer Solutions.—Comparing Equations (14) and (15), one finds

$$\gamma = 4 \qquad \alpha = 1$$

Four possible solutions to Equation (14) are

$$u_1 = {}_1F_1(1,4,z)$$
$$u_2 = z^{-3}{}_1F_1(-2,-2,z)$$
$$u_3 = U(1,4,z)$$
$$u_4 = z^{-3}U(-2,-2,-z)$$

where ${}_1F_1(\alpha,\gamma,x)$ is the confluent hypergeometric function and $U(\alpha,\gamma,x)$ is its associated function. Any two of these four solutions are linearly independent but any three solutions are linearly dependent. By means of special transformations there are four alternate ways of writing the solutions u_1, u_2, u_3, and u_4 (Slater, 1950). These forms are not of interest in the present discussion but they do bring one's supply of functions, from which to choose the two linear independent solutions, to eight. Two of the solutions, u_2 and u_3, can be written in closed form. Since $\alpha = \gamma$ in u_2, (Erdelyi, *et. al.*, 1953)

$${}_1F_1(-2,-2,z) = e$$

Then

$$u_2 = \frac{1}{z^3} ez$$

and

$$G_1 = \frac{2\zeta}{z^2} \quad (16)$$

where the subscript is used to identify a specific solution of the homogeneous equation (5).

That u_3 is in closed form can be seen from the integral representation (Jeffreys and Jeffreys, 1950).

$$U(\alpha,\gamma,x) = \frac{1}{(\alpha-1)!} x^{1-\gamma} \int_0^\infty \frac{e^{-\mu} \mu^{\alpha-1}}{(x+\mu)^{\alpha-\gamma+1}} d\mu \qquad R(\alpha) > 0$$

Hence

$$U(1,4,z) = \frac{1}{z} + \frac{2}{z^2} + \frac{2}{z^3}$$

and

$$G_2 = 2\zeta \left[1 + \frac{2}{z} + \frac{2}{z^2} \right] \quad (17)$$

It is well to note at this point that G_2 is well behaved at $r = \infty$ but not at $r = 0$ while G_1 becomes infinite both at $r = 0$ and $r = \infty$. A solution which is well behaved at $r = 0$ but not at $r = \infty$ is based upon u_1, i.e.

$$G_3 = 2\zeta z\, {}_1F_1(1,4,z) \quad (18)$$

Expansion of the exponential in Equation (16) allows one to verify that

$$G_3 = \frac{1}{6} \left[G_1 - \frac{1}{2} G_2 \right] \quad (19)$$

It is always possible to find a pair of solutions such that one behaves well at zero and the other at ∞. In problems involving Kummer's equation this is usually accomplished by selecting a solution based upon ${}_1F_1(\alpha,\gamma,x)$ (well behaved at $x = 0$) and by selecting one based upon $U(\alpha,\gamma,x)$ (well behaved at $x = \infty$). For informational purposes the limiting form of these solutions in the region of interest follow (Jeffreys and Jeffreys, 1950).

$$r \sim 0 \quad {}_1F_1(\alpha,\gamma,x) = 1 + \frac{\alpha}{\gamma}x + O(x^2)$$

$$r \sim \infty \quad U(\alpha,\gamma,x) \sim x^{-\alpha}$$

$$\left(1 + \frac{\alpha(\gamma-\alpha-1)}{x} + O\left(\frac{1}{x^2}\right)\right)$$

The Whittaker Solutions.—A similar treatment to that just given can be made for solutions based upon Whittaker's standard form. The solutions are called Whittaker functions. For example, after comparing Equations (11) and (12) to obtain the values of k and m, one can write

$$w_1 = M_{1,3/2}(z)$$

$$w_2 = M_{1,-3/2}(z)$$

$$w_3 = W_{1,3/2}(z)$$

$$w_4 = W_{-1,3/2}(-z)$$

The solution for w_3 is given by the integral representation (Whittaker and Watson, 1952).

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z}z^k}{\Gamma(\frac{1}{2} - k + m)}$$

$$\int_0^\infty t^{-k-\frac{1}{2}+m}(1+t/z)^{k-\frac{1}{2}+m}e^{-t}dt$$

The result, for the current problem, is

$$w_3 = e^{-\frac{1}{2}z}\, z\left[1 + \frac{2}{z} + \frac{2}{z^2}\right]$$

which leads directly to the G_2 given by Equation (17).

The solution, for w_4 is found from the asymptotic expansion (Whittaker and Watson, 1952).

$$W_{-k,m}(-z) \sim (-z)^{-k}e^{\frac{1}{2}z}$$

$$\left[1 - \frac{m^2 - (k + \frac{1}{2})^2}{1!z} + \dots\right]$$

For the current problem this series terminates, giving

$$w_4 = -\frac{1}{z}e^{\frac{1}{2}z}$$

which leads directly to G_1 given by Equation (16). The difference in sign does not alter any of the properties discussed in succeeding sections so long as the sign usage is consistent.

Another set of four functions are obtained by replacing k by —k and z by —z. Some books Whittaker and Watson, 1952, and Buchholz, 1953) base their entire treatment on Whittaker functions, while others, e.g. Slater (1960), use both treatments. The latter volume gives the pertinent interrelations. With either treatment, one arrives at the same solutions to the homogeneous equation, (Slater, 1960).

The preceding material has been given in detail, since differential equations whose solutions are confluent hypergeometric functions occur frequently in radial quantum mechanical problems. The texts referred to provide excellent starting points for the reader in obtaining solutions to a particular problem.

Variation of Constants

One method for the solution of an inhomogeneous differential equation is the variation of constants (Burington and Torrance, 1939, p. 361 et. seq.). Basically, the idea for a second order linear differential equation is to first write the equation such that the coefficient of the second derivative is unity, e.g.

$$g''(r) + \left(\frac{2}{r} - 2\right)g'(r) - \frac{2}{r^2}g(r) = -2\varepsilon r \qquad (20)$$

The general solution of the homogeneous equation (Slater, 1960) associated with Equation (20) is a linear combination of two linearly independent solutions of Equation (5). Such a solution is called the *complementary function*. To obtain the general solution of Equation (20), a *particular integral* must be added to the complementary function.

The particular integral is assumed to be a sum of two independent solutions of the homogeneous equation. However, the combining coefficients are themselves functions of the dependent variable. The analogy to the homogeneous solution as a linear combination (but with constant coefficients) of solutions gives rise to the contradiction in terms which entitle this section.

To illustrate the method, the solutions given by Equations (16) and (17) are used. The particular integral is

$$P(G_1,G_2) = C_1(z)G_1 + C_2(z)G_2 \qquad (21)$$

where

$$C_1(z) = -\int \frac{G_2 I}{W}\,dz$$

$$C_2(z) = \int \frac{G_1 I}{W}\,dz$$

$$W(G_1,G_2) = G_1G_2' - G_1'G_2$$

and I is the inhomogeneity of Equation (20). (For other than second order equations, Burington and Torrance (1939) suggests a general method). For the sake of notational simplicity, the use of the variable z is maintained.

For this problem, one finds

$$W(G_1,G_2) = -\frac{4\zeta^4 e^z}{z^2}$$

$$C_1(z) = \frac{e^{-z}}{\zeta^4}\left[\frac{z^3}{8} + \frac{5z^2}{8} + \frac{3z}{2} + \frac{3}{2}\right]$$

$$C_2(z) = \frac{z^2}{16\zeta^4}$$

$$P = \frac{1}{\zeta^3}\left[\frac{z^2}{8} + \frac{z}{2} + \frac{3}{2} + \frac{3}{z} + \frac{3}{z^2}\right]$$

Then

$$g = c_1\left[\frac{2\zeta e^z}{z^2}\right] + c_2 2\zeta\left[1 + \frac{2}{z} + \frac{2}{z^2}\right] + \frac{1}{\zeta^3}\left[\frac{z^2}{8} + \frac{z}{2} + \frac{3}{2} + \frac{3}{z} + \frac{3}{z^2}\right]$$

Since this is not a purely mathematical exercise, one must also ask about the boundary conditions. In particular, the solution for Ψ^1 (Equation (2)) must be finite at both $r = 0$ and $r \to \infty$. This will not be true for $r \to 0$ unless, in the radial function g, $c_1 = 0$ and $c_2 = -3/4\zeta^4$. The decreasing exponential in Ψ° is not sufficient to "kill" the e^z term at $r \to \infty$, but having already chosen $c_1 = 0$, this problem is of no import. Using these values for c_1 and c_2

$$g = \frac{1}{\zeta^3}\left[\frac{z^2}{8} + \frac{z}{2}\right]$$

It should also be pointed out that one will obtain the same solution, Equation (22) if one starts with another pair of solutions of the homogeneous equation (5). For example, if instead of Equation (21) one takes

$$P(G_3,G_2) = C_3(z)G_3 + C_4(z)G_2$$

where G_3 is given by Equation (19), then it can be shown by substitution that

$$W_1(G_3,G_2) = G_3G_2' - G_3'G_2 = \frac{1}{6}W(G_1,G_2)$$

$$C_3(z) = 6C_1(z)$$

$$C_4(z) = C_2(z) + \frac{1}{2}C_1(z)$$

$$P(G_3,G_2) = P(G_1,G_2)$$

Thus, one obtains the same particular integral as before. Although the complementary function is different, the linear constants must again be chosen so as to fit the boundary conditions. Once this is done, one again obtains Equation (22).

Green's Function

The Green's function method (Sagan, 1961) is similar to the method of variation of constants described in the preceding section. This method has particular appeal to physicists while the variation of constants is usually described in books by mathematicians. As will be seen, the principal difference lies in the point of insertion of the boundary conditions. In the method just described, one applied the boundary conditions to the general solution in order to determine the two integration constants. In using the Green's function, the boundary conditions are taken in account during formulation of the problem.

The first step in this method is to write the differential equation in *self-adjoint* form (Sagan, 1961). That is, the equation must be of the form

$$\left\{ \frac{d}{dx}\left[p(x)\frac{d}{dx} \right] + q(x) \right\} y = -I(x) \qquad (23)$$

Equation (3) cannot be put into this form directly; but after making the substitutions, given in Equation (8) and $z = 2\zeta r$, and rearranging, one obtains (corresponding to the homogeneous Equation (11))

$$w'' + w\left[-\frac{1}{4} + \frac{1}{z} - \frac{2}{z^2} \right] = -\frac{z^2}{8\zeta^4}e^{-\frac{1}{2}z}$$

or

$$\left\{ \frac{d}{dz}\left[1\frac{d}{dz} \right] + \left[-\frac{1}{4} + \frac{1}{z} - \frac{2}{z^2} \right] \right\} w = -\frac{z^2e^{-\frac{1}{2}z}}{8\zeta^4} \qquad (24)$$

which is in the self-adjoint form with

$$P(z) = 1$$

$$q(z) = -\frac{1}{4} + \frac{1}{z} - \frac{2}{z^2}$$

$$I(z) = \frac{z^2e^{-\frac{1}{2}z}}{8\zeta^4}$$

The Green's function $G(x, \xi)$ for the standard form, Equation (23), is

$$G(x,\xi) = \begin{cases} -\dfrac{y_1(x)y_2(\xi)}{P(\xi)W[y_1y_2]} & x < \xi \\[2ex] -\dfrac{y_1(\xi)y_2(x)}{P(\xi)W[y_1,y_2]} & \xi \leq x \end{cases}$$

where y_1 (x) and y_2 (x) are two linearly independent solutions of the corresponding homogeneous equation of Equation (23) in the regions, $x < \xi$ and $\xi \leq x$, respectively; W $[y_1,y_2]$ is the Wronskian, $y_1(\xi)y'_2(\xi) - y'_1(\xi)y_2(\xi)$; and ξ is a dummy integration variable. Then the solution of the inhomogeneous Equation (23) is given (Sagan, 1961) by

$$y(x) = \int_a^b G(x,\xi)\ I\ (\xi)\, d\xi$$

where a and b are the end points of the region specified by the boundary conditions.

Since the homogeneous equation associated with Equation (24) is just Equation (11), the solutions y_1 (z) and y_2 (z) are simply Whittaker functions. However, one must be careful to use those functions which behave properly in the appropriate ranges of integration.

As was shown in Section c, two immediate solutions of Equation (11) are

$$w_3(z) = W_{1,3/2}(z) = e^{-\frac{1}{2}z}z \left[1 + \frac{2}{z} + \frac{2}{z^2} \right]$$

$$w_4(z) = W_{-1,3/2}(-z) = -\frac{e^{\frac{1}{2}z}}{z}$$

corresponding to the solutions obtained in Equations (17) and (16), respectively. However, this pair of solutions is not suitable since $W_{-1,3/2}(-z)$ does not behave well at either $r = 0$ or $r = \infty$. Therefore, it is necessary to find the proper linear combination which will behave correctly at $r = 0$ (since in this case, $W_{1,3/2}(z)$ behaves well at $r = \infty$). As is easily verified, by expanding the exponentials, the proper combination is

$$w_5(z) = \frac{1}{2}W_{1,3/2}(z) + W_{-1,3/2}(-z) = \frac{1}{2}w_3 + w_4$$

$$w_3(z) = W_{1,3/2}(z)$$

Except for a numerical factor, which does not affect the problem, the solution $W_5(z)$ corresponds to the combination in Equation (19).

For the present problem, the solution of Equation (24) in these terms is

$$w(z) = -w_3(z)\int_0^{\xi} \frac{w_5(\xi)I(\xi)}{P(\xi)W[w_5,w_3]}d\xi - w_5(z)\int_{\xi}^{\infty} \frac{w_3(\xi)I(\xi)}{P(\xi)W[w_5,w_3]}d\xi \quad (25)$$

Although this example leads to an easily derived Wronskian, it should be pointed out that Slater (1960) tabulates Wronskians for various pairs of Kummer functions and for various pairs of Whittaker functions. For this problem

$$W[w_5,w_3] = 1$$

$$I = \frac{z^2e^{-\frac{1}{2}z}}{8\zeta^4}$$

$$P(\xi) = 1$$

The two integrals are

$$\int_0^z \left[\frac{e^{-\frac{1}{2}\xi}}{2}\left(\xi + 2 + \frac{2}{\xi}\right) - \frac{e^{-\frac{1}{2}\xi}}{\xi}\right]\left[\frac{\xi^2e^{-\frac{1}{2}\xi}}{8\zeta^4}\right]d\xi = -\frac{1}{16\zeta^4}[z^2 + e^{-z}(z^3 + 5z^2 + 12z + 12) - 12] \quad (26a)$$

$$\int_z^{\infty}\left[e^{-\frac{1}{2}\xi}\left(\xi + 2 + \frac{2}{\xi}\right)\right]\left[\frac{\xi^2e^{-\frac{1}{2}\xi}}{8\zeta^4}\right]d\xi = \frac{1}{8\zeta^4}\left[e^{-z}(z^3 + 5z^2 + 12z + 12)\right] \quad (26b)$$

Substituting Equations (26) into Equation (25) gives

$$W(z) = \frac{e^{-\frac{1}{2}z}}{8\zeta^4}\left[\frac{z^3}{2} + 2z^2\right]$$

and applying the inverse of the transformation of Equation (8) [in terms of the new variable $z = 2\zeta r$], once again one obtains

$$g = \frac{1}{\zeta^3}\left[\frac{z^2}{8} + \frac{z}{2}\right] \quad (22)$$

Method of Undetermined Coefficients

There exists still another method of finding a particular integral, the method of assuming a series solution (Ince, 1944) and determining the unknown coefficients of the individual terms in the series. Clearly, in a problem such as the one under discussion, the series will terminate after a finite number of terms.

In order to solve Equation (3), a series solution in z is assumed (after the appropriate transformation to the new variable $z = 2\zeta r$). However, this corresponds to an expansion in a Taylor series about zero which normally implies that one has to worry about whether the point of expansion (zero in this example) is an *ordinary point* or a *singular point* (Ince, 1944). Since in this problem the series is finite, no question of convergence arises and there is no need to worry about the fact that $z = 0$ is really a singular point of Equation (3).

Transforming Equation (3) to an equation in the variable z gives

$$g''(z) + \left(\frac{2}{z} - 1\right)g'(z) - \frac{2}{z^2}\,g(z) = - \frac{z}{4\zeta^3} \qquad (27)$$

The assumed solution is

$$P = Az^3 + Bz^2 + Cz + D \qquad (28)$$

This assumption is reasonable since it contains the information that the particular integral is a finite polynomial and further, after two differentiations, the highest power of z is the highest power in the inhomogeneity.

Substituting Equation (28) into Equation (27) and equating the coefficients of like powers of z leads to the following system of equations

$$-3A = 0$$

$$10A - 2B = -\frac{1}{4\zeta^3}$$

$$4B - C = 0$$

$$-2D = 0$$

whose solution is

$$A = 0$$

$$B = \frac{1}{8\zeta^3}$$

$$C = \frac{1}{2\zeta^3}$$

$$D = 0$$

Therefore, substituting these coefficients in Equation (28), yields

$$P = \frac{1}{\zeta^3}\left[\frac{z^2}{8} + \frac{z}{2}\right]$$

Once again it is necessary to obtain a complete solution by adding a complementary function and evaluating the constants of integration. Then, constructing the complementary function from Equations (17) and (19), for example, one obtains

$$g = c_1G_2 + c_2G_3 + \frac{1}{\zeta^3}\left[\frac{z^2}{8} + \frac{z}{2}\right]$$

Since g must behave properly at $r = 0$ and $r = \infty$, then $c_1 = c_2 = 0$. The solution for g is that previously found

$$g = \frac{1}{\zeta^3}\left[\frac{z^2}{8} + \frac{z}{2}\right]$$

Summary

Three methods of finding a particular integral have been demonstrated and some comments have been made concerning their differences. Since the final answer is the same, the choice between methods usually lies in which method entails the least effort.

Unlike the other methods, the method of undetermined coefficients does not demand knowledge of solutions to the homogeneous equation. If one is fortunate enough to have a problem whose solution is a finite polynomial or an easily recognizable series, this method avoids the work of obtaining complementary functions which are of no practical interest.

A comparison is made of the two sets of solutions arising from the different standard forms of differential equations whose solutions are confluent hypergeometric functions. Facility in the use of confluent hypergeometric functions is a great aid in solving many of the radial problems arising in quantum mechanics. This is particularly true in those cases where the method of undetermined coefficients provides an inconsistent set of algebraic equations and it is necessary to obtain the complementary function. Finally, it should be noted that in many cases the confluent hypergeometric solutions are infinite series which will be inconvenient in determining the particular integral, but the methods are still valid.

Acknowledgment

This research was supported by the National Aeronautics and Space Administration Grant NsG-275-62 (4180) while the author was on leave (1963-4) from Southern Illinois University, Carbondale, Illinois. The encouraging assistance of Prof. Saul T. Epstein whose probing questions led to the comparisons made in this report are gratefully acknowledged. Thanks are also accorded to Prof. W. Byers Brown for helpful discussions during the development of this formalization.

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