

TEMPERATURE PROFILES GENERATED BY A SOURCE TERM IN MOTION:  
A NON-LINEAR HEAT EQUATION

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ABSTRACT: Applications in welding engineering require precise analytic models for temperature transmission through planar work pieces in order to treat weldments for exotic materials such as titanium. The current state of the art views linear quasi-stationary problems for planar heat flow in workpieces whose physical parameters such as thermal conductivity are functions of position only. This investigation is an analytic approach to the solution of a non-linear planar heat flow problem generated from a distributional heat source moving along a smooth curve at constant speed. The quasi-stationary condition is not required and the thermal properties of the workpiece are functions of position and temperature. Thus, the problem is non-linear and time dependent. The solution follows by applying D'Alembert's transformation to the heat equation to generate a formally hyperbolic PDE in the complex-plane that is solved in equivalent form as a non-linear Volterra's integral equation. This produces an analytic solution in the form of a rapidly converging series.

INTRODUCTION

Modern Technology requires the fabrication of a large number of metals, both familiar and exotic, under a variety of conditions. Mathematical studies of the heat flow process in a workpiece upon welding it are a valuable complement to experimental investigations because they are a significant aid in ascertaining the the more important parameters in determining the final quality of a weld. Such studies allow the construction of a framework within which to place experimental results.

According to Rosenthal [12] and Myers [10], interest in the study of the heat flow in a material upon the application of a moving heat source was originally motivated by electric arc welding. Rosenthal himself was the first investigator to apply the heat flow equation with a moving heat source to the study of welding and to obtain analytical solutions for some specific cases. All of the examples considered by Rosenthal involve constant linear

motion. The thermal properties, that is, the thermal conductivity and specific heat of the material were assumed constant and independent of the spatial coordinates, time and temperature. In order to simplify the heat equation, he applied a quasi-stationary condition. This condition was derived from experimental datum in which the temperature distribution around the heat source rapidly becomes time-independent if the region that is being heated is small compared to the entire body.

More complex cases than those of Rosenthal have been investigated by Des Ruisseaux and Zerkle [2], and by Kawashimo, Yamada, and Fujiwara [4]. They obtained analytical solutions for bodies with moving band sources that have cylindrically symmetric thermal properties. The motivation for those studies arose from machining processes.

The increasing need for more realistic mathematical studies of the welding process than Rosenthal's and the advent of the electronic computer have led to the utilization of numerical techniques for the solution of the heat flow equation with a moving heat source. The two main routes taken numerically have been the finite difference and the finite element methods. The numerical techniques allow the utilization of variable thermal properties in the calculations and do not require the imposition of the quasi-stationary condition. Westby [14] was the first investigator to apply the finite difference approach using a computer program. His work was refined and extended by Paley and Hibbert [11]. The application of the finite element approach was originated by Masubucki [9].

Increasing utilization of the computer suggests extension of the work initiated by Rosenthal in obtaining analytical solutions to the moving heat source problem in the more general non-linear cases. This investigation is an analytical approach to the solution of the non-linear two-dimensional heat flow equation with the point heat source moving along an arbitrary curve on the plane at a constant speed. The quasi-stationary condition is not required and the thermal properties of the workpiece are non-constant. Furthermore, the specific heat is analytic function of the spatial coordinates, the temperature, and the time. The thermal conductivity can be an analytic function of the spatial coordinates although it is constant with respect to both temperature and time. More precisely, the specific heat is an implicit function of the time because the time depends explicitly on the temperature and the specific heat is temperature dependent. Also, the two-dimensional planar heat flow models a "very" thin plate or a thick plate where the temperature flow in the z-direction is instantaneous.

A rationale of this extension of the work is that analytical solutions to a particular physical problem often yield greater insight into the physical mechanisms involved than do numerical solutions.

In the work described presently, the global solution for the temperature distribution follows by introducing a hypercomplex coordinate system (in analogy with the classical transformation of the wave equation by d'Alembert) that maps the non-linear heat equation into an equivalent hyperbolic partial differential equation. This equation is integrated to produce a non-linear Volterra integral equation that is solved by successive approximations. For other applications of these methods the reader may consult [1,3,6] and the references therein.

#### DERIVATION OF THE HEAT FLOW EQUATION

Consider the heat flow equation for a two-dimensional plane in its most general form:

$$(1) \quad (\partial E / \partial \mu)(\partial \mu / \partial t) - \vec{\nabla} \cdot [K(x, y, \mu(x, y, t)) \vec{\nabla} \mu] = (\partial Q / \partial t)$$

The factor  $(\partial E(x, y, \mu(x, y, t)) / \partial \mu) = \rho c = \rho(x, y, \mu(x, y, t))c(x, y, \mu(x, y, t))$  is the heat capacity of the material where  $\rho$  is the mass density and  $c$  is the specific heat.

The specific heat of a material for a given temperature  $\mu(x, y, t)$  is defined by  $\partial E$ , the amount of heat required to raise the temperature of a unit mass of that material from  $\mu$  to  $\mu + \delta\mu$  where  $\delta\mu$  is negligible compared to  $\mu$ . The factor  $K = K(x, y, \mu(x, y, t))$  is the thermal conductivity of the material and the term  $(\partial Q(x, y, t) / \partial t)$  represents heat generated by sources internal to the material. In this investigation the internal heat source is a function of position only requiring that  $(\partial Q / \partial t) = 0$ . We proceed by expanding the second term of the left-hand side of equation (1),

$$(2) \quad \rho(x, y, \mu(x, y, t))c(x, y, \mu)(\partial \mu / \partial t) - \vec{\nabla} \cdot [K(x, y, \mu(x, y, t)) \vec{\nabla} \mu]$$

In equation (2)

$$(3a) \quad \vec{\nabla} K = (K_x + K_{\mu} \mu_x) \hat{i} + (K_y + K_{\mu} \mu_y) \hat{j}$$

and

$$(3b) \quad \vec{\nabla} \mu = \mu_x \hat{i} + \mu_y \hat{j},$$

where

$$(3c) \quad K_x = (\partial K / \partial x), \quad K_{\mu} = (\partial K / \partial \mu), \quad \mu_x = (\partial \mu / \partial x), \\ K_y = (\partial K / \partial y), \quad \text{and} \quad \mu_y = (\partial \mu / \partial y)$$

Utilizing equations (3a-c) yields

$$(3c) \quad \nabla \mu \cdot \nabla K = K_x \mu_x + K_y \mu_y + K_{xx} \mu_x^2 + K_{yy} \mu_y^2$$

The equation (2) is now put into the form

$$(4) \quad \rho(x, y, \mu) c(x, y, \mu) \mu_t - K(x, y, \mu) [\mu_{xx} + \mu_{yy}] - [K_x \mu_x + K_y \mu_y] - K_{xx} [\mu_x^2 + \mu_y^2] = 0$$

where  $\mu_{xx} = (\partial^2 \mu / \partial x^2)$ ,  $\mu_{yy} = (\partial^2 \mu / \partial y^2)$  so that utilizing equations (3a-c) in equation (4) gives

$$(5) \quad \rho(x, y, \mu) c(x, y, \mu) (\partial \mu / \partial t) = K(x, y, \mu) \nabla^2 \mu + \nabla K \cdot \nabla \mu + K_{xx} \mu_x^2 + K_{yy} \mu_y^2$$

Let the heat source move at constant speed along a smooth curve in the plane and let  $(\zeta, \eta)$  be the coordinates of a point on the curve that locates the heat source relative to the (stationary) origin. The coordinates  $(\zeta, \eta)$  are functions of the arc length parameters. The coordinates  $(\sigma, \tau)$  are the coordinates for a point in the plane relative to the heat source. Let MC denote a moving coordinate system which has its origin at the heat source with coordinates  $(\zeta, \eta)$  and LC represents a laboratory coordinate system whose origin is stationary. The coordinates  $(x, y)$  in the LC system denote the same point  $(\sigma, \tau)$  in the MC system. The LC and MC systems are related by

$$(6) \quad \sigma = x - \zeta(s), \tau = y - \eta(s).$$

The transformation of the temperature between these coordinate systems is

$$(7a) \quad (\partial \mu / \partial t)_{LC} = (\partial \mu / \partial t)_{MC} + (\partial \mu / \partial \sigma) (\partial \sigma / \partial \zeta) (d\zeta / ds) (ds / dt) + (\partial \mu / \partial \tau) (\partial \tau / \partial \eta) (d\eta / ds) (ds / dt),$$

or since,

$$(7b) \quad (d\sigma / ds) = (\partial \sigma / \partial \zeta) (d\zeta / ds), (d\tau / ds) = (\partial \tau / \partial \eta) (d\eta / ds),$$

$$(7b) \quad (\partial \mu / \partial t)_{LC} = (\partial \mu / \partial t)_{MC} + (\partial \mu / \partial \sigma) (d\sigma / ds) (ds / dt) + (\partial \mu / \partial \tau) (d\tau / ds) (ds / dt).$$

Now, the inverses:

$$(d\sigma / ds) = (ds / d\sigma)^{-1}, (d\tau / ds) = (ds / d\tau)^{-1}$$

exist provided that the curve is smooth

$$(8a) \quad |d\sigma / ds| \leq M, |d\tau / ds| \leq M$$

for some positive constant M. From eq. (6), it is seen that the parameter  $s$  itself is a function of the moving coordinates  $(\sigma, \tau)$ . Then,

$$(8b) \quad (d\sigma/ds) = (1/\dot{s}(\sigma)), \quad (d\tau/ds) = (1/\dot{s}(\tau)),$$

where

$$\dot{s}(\sigma) = (ds(\sigma)/d\sigma), \quad \dot{s}(\tau) = (ds(\tau)/d\tau).$$

Equation (7b) then becomes

$$(9) \quad (\partial\mu/\partial t)_{LC} = (\partial\mu/\partial t)_{MC} + [(\partial\mu/\partial\sigma)(1/\dot{s}(\sigma)) + (\partial\mu/\partial\tau)(1/\dot{s}(\tau))](ds/dt)$$

We require that the speed of the heat source  $|ds/dt|$  be a constant with a value of  $v$ , so that equation (9) becomes

$$(10) \quad (\partial\mu/\partial t)_{LC} = (\partial\mu/\partial t)_{MC} + (v/\dot{s}(\sigma))(\partial\mu/\partial\sigma) + (v/\dot{s}(\tau))(\partial\mu/\partial\tau).$$

Substitution of equation (10) into equation (5) yields

$$(11) \quad \rho(\sigma, \tau, \mu)c(\sigma, \tau, \mu)[(\partial\mu/\partial t)_{MC} + (v/\dot{s}(\sigma))(\partial\mu/\partial\sigma) + (v/\dot{s}(\tau))(\partial\mu/\partial\tau)] \\ = K(\sigma, \tau, \mu)[(\partial^2\mu/\partial\sigma^2) + (\partial^2\mu/\partial\tau^2)] + K_\sigma\mu_\sigma + K_\tau\mu_\tau + K_\mu[\mu_\sigma^2 + \mu_\tau^2].$$

Upon rearrangement equation (11) becomes

$$(12) \quad \rho(\sigma, \tau, \mu)c(\sigma, \tau, \mu)(\partial\mu/\partial t) = K(\sigma, \tau, \mu)[\mu_{\sigma\sigma} + \mu_{\tau\tau}] = [K_\sigma - (v/\dot{s}(\sigma))c(\sigma, \tau, \mu)]\mu_\sigma \\ + [K_\tau + (v/\dot{s}(\tau))c(\sigma, \tau, \mu)]\mu_\tau + K[\mu_\sigma^2 + \mu_\tau^2],$$

where the subscript MC has been dropped. Now, assuming that  $K_\mu \equiv 0$ , so that the thermal conductivity is independent of temperature reduces equation (12) to

$$(13) \quad \rho(\sigma, \tau, \mu)c(\sigma, \tau, \mu)\partial\mu/\partial t = K(\sigma, \tau)[\mu_{\sigma\sigma} + \mu_{\tau\tau}] \\ + [K_\sigma - (v/\dot{s}(\sigma))c(\sigma, \tau, \mu)]\mu_\sigma + [K_\tau + (v/\dot{s}(\tau))c(\sigma, \tau, \mu)]\mu_\tau.$$

Normalization of equation (13) gives

$$(14) \quad d(\sigma, \tau, \mu)(\partial\mu/\partial t) = \\ (\partial^2\mu/\partial\sigma^2) + (\partial^2\mu/\partial\tau^2) + a(\sigma, \tau, \mu)(\partial\mu/\partial\sigma) + b(\sigma, \tau, \mu)(\partial\mu/\partial\tau)$$

where

$$(15a) \quad d(\sigma, \tau, \mu) \equiv (\rho(\sigma, \tau, \mu)c(\sigma, \tau, \mu)/K(\sigma, \tau)),$$

$$(15b) \quad a(\sigma, \tau, \mu) = (1/K(\sigma, \tau)) [K_\sigma - (v/\dot{s}(\sigma))\rho(\sigma, \tau, \mu)c(\sigma, \tau, \mu)],$$

$$(15c) \quad b(\sigma, \tau, \mu) = (1/K(\sigma, \tau)) [K_\tau - (v/\dot{s}(\tau))\rho(\sigma, \tau, \mu)c(\sigma, \tau, \mu)].$$

SOLUTION OF THE HEAT FLOW EQUATION

The real variables  $(\sigma, \tau)$  are now continued as independent complex-variables from which the hyper-complex coordinates

$$(16) \quad \sigma = (z + z^*)/2, \quad \tau = (z - z^*)/(2i)$$

are defined. In the event that  $\sigma$  and  $\tau$  are real, then  $z^*$  is the complex conjugate of  $z$ . In other words,  $z^* = z$  if, and only if,  $(\sigma, \tau)$  are real.

Requiring the specific heat and thermal conductivity to be analytic on  $C^3 = C \times C \times C$  ( $C$  being the complex-plane), equation (14) becomes formally a hyperbolic equation

$$(17a) \quad U_{zz^*} - A(z, z^*, U)U_z - B(z, z^*, U)U_{z^*} = D(z, z^*, U)U_t,$$

in the hyper-complex system with coefficients

$$(17b) \quad A = -1/4(a+ib), B = -1/4(a-ib), D = 1/4d.$$

After rearranging, equation (17a) becomes

$$(18) \quad U_{zz^*} = D(z, z^*, U)U_t + A(z, z^*, U)U_z + B(z, z^*, U)U_{z^*}.$$

Equation (18) is now integrated to produce the (non-linear) Volterra integral equation

$$(19a) \quad U(z, z^*, t) = \int_0^z \int_0^{z^*} H(w, w^*, U) dw^* dw,$$

where

$$(19b) \quad H(w, w^*, U) = D(w, w^*, U)U_t + A(w, w^*, U)U_w + B(w, w^*, U)U_{w^*}.$$

Equation (19a) is to be solved by successive iterations. Thus for  $n \geq 1$ ,

$$\begin{aligned}
 A^{(n-1)} &= A^{(n-1)}(z, z^*, U^{(n-1)}) = A^{(n-1)}(z, z^*), \\
 B^{(n-1)} &= B^{(n-1)}(z, z^*, U^{(n-1)}) = B^{(n-1)}(z, z^*) \\
 (20) \quad D^{(n-1)} &= D^{(n-1)}(z, z^*, U^{(n-1)}) = D^{(n-1)}(z, z^*)
 \end{aligned}$$

In general,  $A^{(0)}$ ,  $B^{(0)}$ ,  $C^{(0)}$  are arbitrary constants. Equation (18) is then solved for  $U^{(0)}$  by the methods of Bergman and Gilbert [9], [10]. However, a natural choice for our problem is  $U^{(0)} = U^{(0)}(z, z^*, t)$ , the ambient temperature profile in the plate.

The iterated form of equation (19a) is, for  $n \geq 1$ ,

$$(21a) \quad U^{(n)}(z, z^*, t) = \int_0^z \int_0^{z^*} H^{(n-1)}(w, w^*, U^{(n-1)}) dw^* dw,$$

where

$$\begin{aligned}
 (21b) \quad & H^{(n-1)}(w, w^*, U^{(n-1)}) = D^{(n-1)}(w, w^*, U^{(n-1)})U^{(n-1)} \\
 & + A^{(n-1)}(w, w^*, U^{(n-1)})U_w^{(n-1)} + B^{(n-1)}(w, w^*, U^{(n-1)})U_{w^*}^{(n-1)}
 \end{aligned}$$

The validity of the iterative approach is dependent on the absolute and uniform convergence of the telescoping series

$$\begin{aligned}
 (22a) \quad (U^{(0)} - U^{(1)}) + \sum_{k=1}^n W^{(k+1)} &= (U^{(0)} - U^{(1)}) + (U^{(1)} - U^{(2)}) + \dots + (U^{(n-1)} - U^{(n)}) \\
 &= U^{(0)} - U^{(n+1)}
 \end{aligned}$$

where

$$(22b) \quad W^{(k+1)} = U^{(k-1)} - U^{(k)}, \quad k = 1, 2, \dots, n$$

The demonstration of the convergence commences with the series

$$(23a) \quad \sum_{n=0}^{\infty} W^{(n+1)} = \sum_{n=0}^{\infty} (U^{(n-1)} - U^{(n)})$$

where

$$(23b) \quad W^{(n+1)} = \int_0^z \int_0^{z^*} [H^{(n+1)}(w, w^*, U^{(n+1)}) - H^{(n)}(w, w^*, U^{(n)})] dw dw^*,$$

and

$$\int_0^z \int_0^{z^*} [H^{(n+1)}(w, w^*, U^{(n+1)}) - H^{(n)}(w, w^*, U^{(n)})] dw dw^*$$

$$= \int_0^z \int_0^{z^*} (F^{(n+1)} + G^{(n+1)} + L^{(n+1)}) dw dw^*,$$

$$F^{(n+1)} = [D^{(n+1)}(w, w^*, U^{(n+1)})U_t^{(n+1)} - D^{(n)}(w, w^*, U^{(n)})U_t^{(n)}]$$

$$G^{(n+1)} = [A^{(n+1)}(w, w^*, U^{(n+1)})U_w^{(n+1)} - A^{(n)}(w, w^*, U^{(n)})U_w^{(n)}],$$

$$L^{(n+1)} = [B^{(n+1)}(w, w^*, U^{(n+1)})U_{w^*}^{(n+1)} - B^{(n)}(w, w^*, U^{(n)})U_{w^*}^{(n)}]$$

Let

$$M = \max\{|D^{(n+1)}|, |D^{(n)}|, |A^{(n+1)}|, |A^{(n)}|, |B^{(n+1)}|, |B^{(n)}|\}, n \geq 1$$

As we seek the analytic solution, the constant is finite for the arguments of the functions on compact simply connected sets in  $C^3$ . From equation (23b),

$$|W^{(n+1)}| \leq \int_0^z \int_0^{z^*} [ |F^{(n+1)}| + |G^{(n+1)}| + |L^{(n+1)}| ] dw dw^*, n \geq 0$$

Now consider the first term in the integrand of equation (24), (reducing the index  $n$  by 1)

$$|F^{(n+1)}| \leq \int_0^z \int_0^{z^*} |F^{(n)}| dw dw^*, n \geq 0.$$

Let

$$K_1 = \max_t |U^{(1)} - U^{(0)}|$$

$$|F^{(2)}| \leq \int_0^z \int_0^{z^*} |F^{(1)}| dw dw^* \leq MK_1 \int_0^z \int_0^{z^*} dw dw^* = MK_1 z z^*.$$

Similarly,

$$|F^{(3)}| \leq \int_0^z \int_0^{z^*} |F^{(2)}| dw dw^* \leq MK_1 \int_0^z \int_0^{z^*} w w^* dw dw^* = MK_1 \frac{z^2}{2!} \frac{(z^*)^2}{2!}$$

so that by induction

$$|F(n)| \leq MK_1 \frac{z^n (z^*)^n}{n!n!}, \quad n \geq 1,$$

Likewise:

$$|G(n)| \leq MK_2 \frac{z^n (z^*)^n}{n!n!}, \quad n \geq 1,$$

$$|L(n)| \leq MK_3 \frac{z^n (z^*)^n}{n!n!}, \quad n \geq 1,$$

Where  $K_2$  and  $K_3$  are defined similarly to  $K_1$ .

Substituting the results of equations (25a-c) into equation (24) yields

$$\begin{aligned} |W(n+1)| &\leq M(K_1 + K_2 + K_3) \int_0^z \int_0^{z^*} \frac{w^{(n+1)}(w^*)^{n+1}}{(n+1)!(n+1)!} dw^*dw \\ &= M(K_1 + K_2 + K_3) \frac{z^{n+2}(z^*)^{n+2}}{(n+2)!(n+2)!}, \quad n \geq -1. \end{aligned}$$

Thus, the series

$$\left| \sum_{n=0}^{\infty} W(n+1) \right| \leq \sum_{n=0}^{\infty} |W(n+1)| \leq M(K_1 + K_2 + K_3) \sum_{n=0}^{\infty} \frac{|z|^{n+2} |z^*|^{n+2}}{(n+2)!(n+2)!}$$

is absolutely and uniformly convergent for all compact subsets of  $C^2 X[0, t_0]$ .

Consequently, the telescoping series equation (22a) is absolutely and uniformly convergent on simply connected compact sets of  $C^2 X[0, t_0]$ . This means that the function

$$(26) \quad U(z, z^*, t) = U(0)(z, z^*, t) - \lim_{n \rightarrow \infty} U^{(n+1)}(z, z^*, t)$$

represents the analytic solution on compact sets of  $C^2 X[0, t_0]$ . In other words, a global solution. The error in truncating the series equation (22a) for an approximate solution of  $(n+1)$  terms is less than  $O(R^n (R^*)^n / ((n+1)!)^2)$  for  $|z| < R$ ,  $|z^*| < R^*$ . Moreover, the complex valued solution in real coordinates  $(\sigma, \tau)$  is recovered by inverting the transformation (16) as

$$z = \sigma + i\tau, \quad z^* = \sigma - i\tau$$

The real solution is the real part of the complex-valued solution.

Let us now consider an elementary example of this technique. The heat source is travelling at constant speed  $v$  along the straight line:  $\zeta = \alpha s$ ,  $\eta = \beta s$  where  $\alpha$  and  $\beta$  are constants. The density  $\rho$ , specific heat  $c$  and the thermal conductivity  $K$  are constants. In real coordinates, the heat flow equation is

$$(27) \quad \rho c u_t = K[\mu_{\sigma\sigma} + \mu_{\tau\tau}] + cv(\alpha\mu_{\sigma} + \beta\mu_{\tau}).$$

The coefficients  $A(n)$ ,  $B(n)$ ,  $D(n)$  are constants for all values of  $n$  and are given the constant values  $\lambda$ ,  $m$ ,  $j$ , respectively, where

$$(28) \quad \lambda = (1/4v)(\rho c/K)(\alpha + \beta i), \quad m = (-1/4v)(\rho c/K)(\alpha - \beta i), \quad j = (1/4)(c/K).$$

In equation (26), the relationship between  $U^{(n)}$  and  $U^{(n+1)}$  is given by (neglecting the time dependence)

$$(29) \quad U^{(n+1)}(z, z^*) = (\ell L_1 + mL_2)U^{(n)}(z, z^*),$$

where  $L_1$  and  $L_2$  are operators such that

$$(30) \quad \begin{aligned} L_1 U^{(n)}(z, z^*) &= \int_0^{z^*} U^{(n)}(z, w^*) dw^*, \\ L_2 U^{(n)}(z, z^*) &= \int_0^z U^{(n)}(w, z^*) dw \end{aligned}$$

Utilization of equation (29) gives

$$(31) \quad U^{(n+1)} = (\ell L_1 + mL_2)^{n+1} U^{(0)}.$$

Take  $\ell = m$ , and  $r = \ell(L_1 + L_2)$ . Then

$$(32a) \quad U(z, z^*) = U^{(0)}(z, z^*) - \lim_{n \rightarrow \infty} r^{n+1} U^{(0)}(w, w^*), \text{ or}$$

$$(32b) \quad U(z, z^*) = U^{(0)}(z, z^*) - \lim_{n \rightarrow \infty} (-1 + (1-r)(1+r+r^2 + \dots + r^n)) U^{(0)}(w, w^*).$$

Consider the case where the temperature distribution in the complex-plane is the Gaussian distribution

$$u^{(0)}(\sigma, \tau) = T_0 e^{-(\sigma^2 + \tau^2)/\varepsilon},$$

where  $\varepsilon$  is a parameter that defines the "diffuseness" of the heat source.

The zero-order approximation is  $u(\sigma, \tau)_{n=0} = T_0 e^{-(\sigma^2 + \tau^2)/\varepsilon} - T_0 \ell \sqrt{\varepsilon \pi}$

$$\left[ e^{-\tau^2/\varepsilon} \Phi_0\left(\frac{\sigma\varepsilon}{\frac{2}{\varepsilon}}\right) + e^{-\sigma^2/\varepsilon} \Phi_0\left(\frac{\tau}{\frac{2}{\varepsilon}}\right) \right]$$

where  $\Phi_0(\xi)$  is the probability integral  $\frac{1}{\sqrt{2\pi}} \int_0^\xi e^{-t^2/2} dt$ .

Runs have been made for aluminum and titanium under various conditions. The "bow wave" phenomena, a heat flow analogy to the "sonic boom", which was noted in previous experiments and calculations [12] was observed.

This phenomenon demonstrates that the peak temperature on the plane occurs ahead of the heat source. For aluminum:

$$\rho=2.70\text{g/cc}(20^\circ\text{C}), c=0.215\text{cal/g}^{-10}\text{K}^{-1}(25^\circ\text{C}), K=2.37\text{watts/cm }^\circ\text{C}(25^\circ\text{C}).$$

For titanium:

$$\rho=4.5\text{g/cc}, c=0.125\text{cal/g }^\circ\text{C}(25^\circ\text{C}), K=0.2\text{watts/cm }^\circ\text{C}(25^\circ\text{C}). [12]$$

#### CONCLUDING REMARKS

The work presented herein constitutes a generalization of the work of Rosenthal [12] for the heat flow in a material where the heat source is moving with constant speed in a straight line on a plane. In this investigation, the results do not require the assumption of a quasi-stationary condition. This allows the investigation of transient processes while a workpiece is being welded. The specific heat and thermal conductivity are not required to be constant although the thermal conductivity is required to be independent of the temperature. The heat source is allowed to move along any smooth curve in the plane.

As noted earlier, the results presented herein apply to "very" thin plates. These results can also be meaningfully interpreted for heat flows with cylindrical symmetry in the mathematical sense. Consider a solid extending to  $\pm\infty$  in the z direction. In this event, all of the functions  $\Phi(x,y,t) = \Phi(x,y,0,t) = \Phi(x,y,z,t)$  assume constant values along lines parallel to the 0-axis.

Further, it must be noted that our results are valid only when the velocity of the heat source is less than the rate of heat conduction. This means that these results are a function of the ratio of the velocity of the heat source to the thermal conductivity. Analytic solutions may not be possible for the case when the velocity of the heat source is greater than the rate of heat conduction because of the apparent singularity that builds up. The velocity of the propagation of the heat as a "bow wave." conduction is given by the square root of the ratio of the thermal diffusivity and relaxation time [7]. For a metal, such as aluminum, this relaxation time is of the order of  $10^{-11}\text{sec}$  [13]. For aluminum, at  $0^\circ\text{C}$ , the thermal diffusivity is approximately  $10^{-3}\text{ft}^2/\text{sec}$  [14]. The velocity of propagation is then approximately  $10^4\text{ft/sec}$  a velocity which is far greater than the maximum anticipated speed of the heat source of  $3\text{ft/sec}$  in electric arc welding. It is such welding which is the motivation for this study.

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