A FIXED-POINT INDEX FOR SYMMETRIC PRODUCT MAPS ON EUCLIDEAN SPACES

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ABSTRACT. The author shows that the classical Holpf-Lefschetz-Leray fixed-point index can be extended to include a certain type of "multiple-valued" function. Specifically, for continuous maps from an open subset V of a Euclidean space E into the n-fold symmetric product $E^{(n)}$ of E (with compact fixed-point set) an integer-valued index is defined. Properties analogous to the classical (n=0) case are established, including the relationship to the Lefschetz number of a map. The elements of $E^{(n)}$ are unordered sub-sets of E having n elements, with repetitions allowed.

INTRODUCTION

For a given positive integer n and topological space E, $E^{(n)}$ will denote the n-fold symmetric product, which is the quotient space of the Cartesian product E^n obtained by identifying two elements if one is obtained from the other by permuting coordinates. A map $f: V \to E^{(n)}$ is said to have <u>multiplicity n</u>. If $x \in V \subseteq E^{(n)}$, then x is <u>fixed</u> under $f: V \to E^{(n)}$ if x is a coordinate of f(x), and K_f will denote the set of fixed points of f.

(0.1) A continuous map $f: V \to E^{(n)}$ is <u>admissible</u> if E is euclidean q-space $\mathbb{R}^q$ for some $q > 0$, V is open in E, and K_f is compact.

The purpose of this article is to define an integer-valued fixed-point-index I(f) for admissible maps f whose non-vanishing guarantees the existence of a fixed point. The development given here follows rather closely that of A. Dold (see [1], VII, 5). The roots of this theory lead back to the monumental work of Leray [3] and Lefschetz (see [8] for historical details).

After the notation and homological machinery is set-up in sections 1, 2, and 3, the index is defined and its homotopy invariance is proved in section 4. In section 5, two additivity formulas are established. The multiplicative property is presented in section 6. Under the assumption that the admissible map factors through a compact subset of $E^{(n)}$, the index can be expressed as a Lefschetz number, as we will show in section 7. A commutativity property is presented in section 8.

(0.2) A source of examples of admissible maps arises from a group $G = \{q_1, \ldots, q_n\}$ of order n with a right operation on a topological space X. Let $h: V \to X\backslash G$ and $k: X \to E$ be continuous, where V is open in $E = \mathbb{R}^q$. For $x \in V$, select $y \in X$ for which $hx = yG \in X\backslash G$. Define $f(x) = [k(yg_1), k(yg_2), \ldots, k(yg_n)] \in E^{(n)}$. If K_f is compact, the map $f: V \to E^{(n)}$ is admissible.

(0.3) Another example is given by the inverse function of a polynomial function $P: \mathbb{C} \to \mathbb{C}$ of degree $n > 0$, over an open set V of $\mathbb{C}$. For $x \in V$, let $f(x) = [z_1, \ldots, z_n]$ be the element of $\mathbb{C}^{(n)}$ giving the solutions to the equation $P(z) = x$. Then $K_f = K_P \cap V$ is finite and f is admissible. More generally, symmetric product maps arise frequently in algebraic geometry.

§1. Symmetric Products and Their Structure Maps. Given a topological space X and $n > 0$, the element of $X^{(n)}$ represented by $(x_1, \ldots, x_n) \in X^n$ will be denoted by $[x_1, \ldots, x_n]$. If $A \subseteq X$, then $A^{(n)} \subseteq X^{(n)}$. Given a map $f: (X, A) \to (Y, B)$, then f induces a map $f^{(n)}: (X^{(n)}, A^{(n)}) \to (Y^{(n)}, B^{(n)})$ given by

$$(1.1) \qquad f^{(n)}[x_1, \ldots, x_n] = [f(x_1), \ldots, f(x_n)]$$

for all $[x_1, \ldots, x_n] \in X^{(n)}$. If $f, \bar{f}: (X, A) \to (Y, B)$ are homotopic, then $f^{(n)}$ and $\bar{f}^{(n)}$ are homotopic also. The correspondence $(\ ,\ )^{(n)}$ given by

$$(1.2) \qquad (X, A) \longmapsto (X^{(n)}, A^{(n)}), \quad f \longmapsto f^{(n)}$$

is a functor on the category of pairs of topological spaces and induces a functor on its homotopy category.

(1.3) The identification map $p_n: X^n \to X^{(n)}$ is an open map for any space X. Consequently, the product map $p_n \times p_m: X^n \times X^m \to X^{(n)} \times X^{(m)}$ is also an identification map, since it is continuous, open, and surjective.

For a space pair (X, A) and positive integers n and m, there is an <u>adjoining map</u>

$$(1.4) \quad \alpha_n{}^m(X, A): (X^{(n)} \times X^{(m)}, A^{(n)} \times A^{(m)}) \to (X^{(n+m)}, A^{(n+m)})$$

given by $\alpha_n{}^m(X, A)([x_1, \ldots, x_n], [y_1, \ldots, y_m]) = [x_1, \ldots, x_n, y_1, \ldots, y_m]$. The adjoining map gives a pairing which is associative and natural. Its continuity may be proved by employing (1.3). More generally, we may speak of a natural <u>adjoining map</u> for positive integers $n_1, \ldots, n_m$ and a space pair (X, A)

$$(1.5) \quad (X^{(n_1)} \times \ldots \times X^{(n_m)}, A^{(n_1)} \times \ldots \times A^{(n_m)}) \to (X^{(n_1 + \ldots + n_m)}, A^{(n_1 + \ldots + n_m)})$$

defined by concatenation.

In particular, if $n_i = n$ for $i = 1, \ldots, m$, the adjoining map

$$(X^{(n)m}, A^{(n)m}) \xrightarrow{\alpha} (X^{(nm)}, A^{(nm)})$$

is equivariant with respect to permutation of the m coordinates. Hence there is induced a natural map

$$(1.6) \quad \beta_n{}^m(X, A): (X^{(n)(m)}, A^{(n)(m)}) \to (X^{(nm)}, A^{(nm)})$$

given by

$$\beta_n{}^m[[x_1{}^1, \ldots, x_n{}^1], \ldots, [x_1{}^m, \ldots, x_n{}^m]] =$$
$$[x_1{}^1, \ldots, x_n{}^1, \ldots, x_1{}^m, \ldots, x_n{}^m].$$

The continuity of β follows from that of α.

For a space pair (X, A) and for positive integers m and n, the diagonal map Δ_m of $X^{(n)}$ into $X^{(n)m}$ induces a map $d_n^{nm}(X, A)\colon (X^{(n)}, A^{(n)}) \to (X^{(nm)}, A^{(nm)})$, also called the <u>diagonal</u>, defined to be the composition $\beta_n{}^m p \Delta_m$:

(1.7)
$$\begin{array}{ccc} (X^{(n)}, A^{(n)}) & \xrightarrow{\Delta_m} & (X^{(n)m}, A^{(n)m}) \\ & & \downarrow p \\ (X^{(mn)}, A^{(mn)}) & \xleftarrow{\beta_n{}^m} & (X^{(n)(m)}, A^{(n)(m)}) \end{array}$$

where p is the canonical projection.

A <u>composition</u> $*$ of maps may be defined as follows. Given maps $f\colon (X, A) \to (Y^{(n)}, B^{(n)})$ and $g\colon (Y, B) \to (Z^{(m)}, C^{(m)})$ define $g*f\colon (X, A) \to (Z^{(mn)}, C^{(mn)})$ to be the composition

(1.8)
$$\begin{array}{ccccc} (X, A) & \xrightarrow{f} & (Y^{(n)}, B^{(n)}) & \xrightarrow{g^{(n)}} & (Z^{(m)(n)}, C^{(m)(n)}) \\ & & & & \downarrow \beta_m{}^n(Z, C) \\ & & & & (Z^{(mn)}, C^{(mn)}) \end{array}$$

where β is defined in (1.5). The composition $*$ is associative.

The function $f\colon V \to E^{(n)}$ of example (0.2) may now be described as the composition $k*\hat{p}*h$ where $\hat{p}\colon X\backslash G \to X$ is given by $\hat{p}(y) = [xg_1, \ldots, xg_n]$ for any x for which $y = xG$.

Given maps $f: (X, A) \to (Y^{(n)}, B^{(n)})$ and $g: (X, A) \to (Y^{(m)}, B^{(m)})$, the adjunction $f \vee g: (X, A) \to (Y^{(n+m)}, B^{(n+m)})$ is defined as the composition

$$(1.9) \qquad \begin{array}{ccc} (X, A) & \xrightarrow{f \times g} & (Y^{(n)} \times Y^{(m)}, B^{(n)} \times B^{(m)}) \\ & & \Big\downarrow \alpha_n{}^m(Y, B) \\ & & (Y^{(n+m)}, B^{(n+m)}) \end{array}$$

where α is defined in (1.4).

The adjunction of maps is commutative and associative. By iteration, there is an adjunction $f_1 \vee \ldots \vee f_m$ of maps $f_i: (X, A) \to (X^{(n_i)}, A^{(n_i)})$ for any $m > 0$, which may be obtained also by $\alpha(f_1 \times \ldots \times f_m)$ with α as in (1.5).

Given pairs (Z, C) and (Y, D) and positive integers n and m, there is a product map

$$(1.10) \quad \pi: (Z^{(n)} \times Y^{(m)}, C^{(n)} \times D^{(m)}) \to ((Z \times Y)^{(nm)}, (C \times D)^{(nm)})$$

given by

$$\pi([z_1, \ldots, z_n], [y_1, \ldots, y_m]) = [(z_i, y_j)]_{1 \le i \le n,\ 1 \le j \le m}.$$

In other words, $\pi(z, y)$ is the unordered nm-tuple consisting of pairs $(z_i, y_j) \in Z \times Y$ with $1 \le i \le n$, $1 \le j \le m$. The map π' defined on the cartesian product in a corresponding manner is obviously continuous. Hence using (1.3) again in the commutative diagram

$$\begin{array}{ccc} Z^n \times Y^m & \xrightarrow{\pi'} & (Z \times Y)^{nm} \\ \downarrow{\scriptstyle p_n \times p_m} & & \downarrow{\scriptstyle p_{nm}} \\ Z^{(n)} \times Y^{(m)} & \xrightarrow{\pi} & (Z \times Y)^{(nm)}, \end{array}$$

one proves the continuity of π.

(1.11) If $f: (X, A) \to (Z^{(n)}, C^{(n)})$ and $g: (W, D) \to (Y^{(n)}, D^{(m)})$ are maps, then the <u>symmetric product of f and g</u> is $\pi(f \times g)$, where $(f \times g)(x, w) = (f(x), g(w))$, as indicated:

$$(X \times W, A \times D) \xrightarrow{f \times g} (Z^{(n)} \times Y^{(m)}, C^{(n)} \times D^{(m)}) \xrightarrow{\pi}$$
$$((Z \times Y)^{(nm)}, (C \times D)^{(nm)})$$

where π is the product map of (1.10).

(1.12) It is not hard to show that if X is Hausdorff, $V \subseteq X$, and $f: V \to X^{(n)}$ continuous, $n > 0$, then the fixed point set K_f is a closed subset of V. Similarly, we establish the following result about homotopies.

(1.13) LEMMA. <u>Let X be Hausdorff $V \subseteq X$, and $F: V \times I \to X^{(n)}$ be continuous. Let $M = \cup_{t \in I} K_{F_t}$. If there is a compact set A such that $M \subseteq A \subseteq V$, then M is compact.</u>

PROOF. Let $W = \{(x, [y_1, \ldots, y_n]) \in X \times X^{(n)} \mid x = y_i$ for some $i = 1, \ldots, n\}$. Using the Hausdorff separation of X, one shows that W is closed in $X \times X^{(n)}$. Let $h: V \times I \to X \times X^{(n)}$ be given by $h(x, t) = (x, F(x, t))$. Then h is continuous; $h^{-1}(W)$ is closed in $V \times I$, and is contained in the compact set $A \times I$. Hence, $h^{-1}(W)$ is compact. Since M is the projection of $h^{-1}(W)$ in V, M is compact.

□

§2. Symmetric Products of Euclidean Spaces. Throughout this section E will denote the euclidean space $\mathbb{R}^q$ for some $q > 0$ and n will denote a positive integer. The functions $s\colon \mathbb{R} \times E^{(n)} \to E^{(n)}$ given by

$$(2.1) \qquad s(t, [x_1, \ldots, x_n]) = [tx_1, \ldots, tx_n]$$

and $D\colon E \times E^{(n)} \to E^{(n)}$ given by

$$(2.2) \qquad D(x, [x_1, \ldots, x_n]) = [x - x_1, \ldots, x - x_n]$$

will be called scalar multiplication and difference, respectively. They are obviously well-defined. We check continuity. The maps $1_{\mathbb{R}} \times p\colon \mathbb{R} \times E^n \to R \times E^{(n)}$ and $1_E \times p\colon E \times E^n \to E \times E^{(n)}$ are identification maps since $1_{\mathbb{R}}$, 1_E, and p are open and surjective. If we let $\bar{s}\colon \mathbb{R} \times E^n \to E^n$ be given by $\bar{s}(t, (x_1, \ldots, x_n)) = (tx_1, \ldots, tx_n)$, then $p\bar{s} = s(1_{\mathbb{R}} \times p)$. Since $\bar{s}$ is continuous and $1_{\mathbb{R}} \times p$ is an identification, s is continuous. Similarly, let $\bar{D}\colon E^n \times E^n \to E^n$ be given by

$$\bar{D}((x_1, \ldots, x_n), (y_1, \ldots, y_n)) = (x_1 - y_1, \ldots, x_n - y_n).$$

In the commutative diagram

$$\begin{array}{ccccc} E \times E^n & \xrightarrow{\Delta \times 1_{E^n}} & E^n \times E^n & \xrightarrow{\bar{D}} & E^n \\ \Big\downarrow 1_E \times p & & & & \Big\downarrow p \\ E \times E^{(n)} & & \xrightarrow{\quad D \quad} & & E^{(n)} , \end{array}$$

where Δ is the diagonal, $p\bar{D}(\Delta \times 1_{E^n})$ is continuous and $1_E \times p$ is an identification. Hence D is continuous.

If $V \subseteq E = R^q$ with inclusion map $i: V \to E$, then for any map $f: V \to E^{(n)}$, the <u>difference</u> $(i - f): V \to E^{(n)}$ is defined to be the composition

(2.3) $$V \xrightarrow{\Delta} V \times V \xrightarrow{i \times f} E \times E^{(n)} \xrightarrow{D} E^{(n)}$$

where Δ is the diagonal map.

The addition $a: E \times E \to E$ of euclidean space $E = \mathbb{R}^q$ may also be carried over to the symmetric product $E^{(n)}$, but at the price of increasing multiplicity. However, the addition can be useful in constructing homotopies as we shall see in section 8. We define <u>addition</u> $A: E^{(n)} \times E^{(n)} \to E^{(n^2)}$ as the composition

(2.4) $$E^{(n)} \times E^{(n)} \xrightarrow{\pi} (E \times E)^{(n^2)} \xrightarrow{a^{(n^2)}} E^{(n^2)}$$

where π is the product map of (1.10), a is vector addition in E, and $a^{(n^2)}$ is its symmetrization (see (1.1)). That is $A([x_1, \ldots, x_n], [y_1, \ldots, y_n]) = [x_i + y_j: 1 \leq i \leq n, 1 \leq j \leq n]$. It follows easily that

(2.5) $$A(0, y) = d_n^{n^2} y \quad \text{and} \quad A(x, 0) = d_n^{n^2} x$$

for $x, y \in E^{(n)}$ where $d_n^{n^2}$ is the diagonal map of (1.7), $0 = [0, \ldots, 0] \in E^{(n)}$.

§3. The trace homology homomorphism. In this section, we will define a natural homomorphism, of degree 0,

$$\mu^n: \ H(X^{(n)}, A^{(n)}) \rightarrow H(X, A) \tag{3.1}$$

where H(-) denotes the singular homology functor with integer coefficients and (X, A) ranges over the category of CW pairs. μ^n will be called the trace homomorphism. We will show that μ^n gives a homomorphism from the exact sequence of the pair $(X^{(n)}, A^{(n)})$ to that of the pair (X, A). We will show that the trace satisfies certain compatibility conditions with respect to the maps α, β, and d introduced in (1.4), (1.6), and (1.7).

Let K be a simplicial set with face maps d_i: $K_m \rightarrow K_{m-1}$ and degeneracy maps s_i: $K_m \rightarrow K_{m+1}$, for $m \geq 0$, $0 \leq i \leq m$. For each $n > 0$, the n-fold (Cartesian) product K^n of K is the simplicial set with $(K^n)_m = (K_m)^n$ for all $m \geq 0$, with faces and degeneracies defined coordinate-wise. The symmetric group gives a simplicial action on K^n by permuting the coordinates. Identifying modulo this group, we obtain $K^{(n)}$, the n-fold symmetric product of K. For $m \geq 0$, $(K^{(n)})_m = (K^n)_m$. An m-simplex of $K^{(n)}$ is an unordered n-tuple $[x_1, \ldots, x_n]$ of m-simplexes of K. The face and degeneracy maps of $K^{(n)}$ are given by

$$(3.3)\qquad \begin{aligned} d_i[x_1, \ldots, x_n] &= [d_i x_1, \ldots, d_i x_n] \\ s_i[x_1, \ldots, x_n] &= [s_i x_1, \ldots, s_i x_n] \end{aligned}$$

for $[x_1, \ldots, x_n] \in K^{(n)}$ and $0 \le i \le m$, and all $n > 0$.

If L is a subcomplex of the simplicial set K, then $L^{(n)}$ may be identified as a subcomplex of $K^{(n)}$.

The geometric realization $|K|$ of a simplicial set K has as elements equivalence classes $|x, u|$ of elements $(x, u) \in K_m \times \Delta_m$, where Δ_m is the standard m-simplex (see [3], p. 55). A simplicial map $f\colon K \to L$ induces a continuous function $|f|\colon |K| \to |L|$ by the rule $|f||x, u| = |fx, u|$.

For every positive integer n and simplicial set K, there is a natural weak homotopy equivalence

$$(3.4)\qquad \rho\colon\ |K^{(n)}| \to |K|^{(n)}$$

given by $\rho|[x_1, \ldots, x_n], u| = [|x_1, u|, \ldots, |x_n, u|]$ where $[x_1, \ldots, x_n] \in (K^{(n)})_m$, $u \in \Delta_m$, (see [7], p. 162, 6.6).

For any space, the geometric realization $|K(X)|$ of its singular complex $K(X)$ is a CW complex which is weakly homotopy equivalent to X via a natural map

$$(3.5)\qquad j\colon\ |K(X)| \to X$$

given by the rule $j|\sigma, u| = \sigma(u)$ for $\sigma \in K_m(X)$, $u \in \Delta_m$ (see [5]). In case X itself is a CW complex, then j is a homotopy equivalence (see [6], VII 24, p. 405).

For any space X, there is a natural simplicial map

$$\theta\colon\ K(X)^{(n)} \to K(X^{(n)})$$

given by

$$\theta[\sigma_1, \ldots, \sigma_n] = p\circ(\sigma_1 \times \ldots \times \sigma_n)\circ\Delta \tag{3.6}$$

where $\sigma_1, \ldots, \sigma_n \in K_m(X)$, Δ is the diagonal of the standard m-simplex into its n-fold cartesian product, and p is the projection of X^n onto $X^{(n)}$.

(3.7) LEMMA. <u>For a CW complex X and $n > 0$, $|\theta(X)|: |K(X)^{(n)}| \to |K(X^{(n)})|$ is a weak homotopy equivalence.</u>

PROOF. The map $j(X)\colon |K(X)| \to X$ of (3.5) is a homotopy equivalence since X is a CW complex. Hence $j(X)^n\colon |K(X)|^n \to X^n$ and $j(X)^{(n)}\colon |K(X)|^{(n)} \to X^{(n)}$ are homotopy equivalences.

Now consider the diagram of continuous maps

$$\begin{array}{ccc} |K(X^{(n)})| & \xrightarrow{j(X^{(n)})} & X^{(n)} \\ \Big\uparrow{\scriptstyle |\theta(X)|} & & \Big\uparrow{\scriptstyle j(X)^{(n)}} \\ |K(X)^{(n)}| & \xrightarrow{\rho(X)} & |K(X)|^{(n)} \end{array} \tag{3.8}$$

where ρ is as in (3.4). One checks that $j(X)^{(n)}\rho(X)|[\sigma_1, \ldots, \sigma_n], u| = [\sigma_1(u), \ldots, \sigma_n(u)] = p\circ(\sigma_1 \times \ldots \times \sigma_n)\circ\Delta(u) = j(X^{(n)})|\theta(X)||[\sigma_1, \ldots, \sigma_n], u|$, for $|[\sigma_1, \ldots, \sigma_n], u| \in |K(X)^{(n)}|$. Hence (3.8) commutes. Then $|\theta(X)|$ must be a weak homotopy equivalence since the remaining three maps of (3.8) are weak homotopy equivalences. $\square$

For any simplicial set K, C(K) will denote the chain complex of K. That is, $C_m(K)$ is the free abelian group on K_m, and $\partial x = \Sigma_i(-1)^i d_i x$ for $x \in K_m$. The homology of C(K) is, by definition, the homology H(K) of K. For a space X, the singular homology H(X) is, by definition, H(K(X)) where K(X) is the singular complex of X.

If $A \subseteq X$, where X is any space, the chain map $\theta(X)_\#$ induced by $\theta(X)$ will induce a chain map

(3.9) $\theta(X, A)_\#\colon\ C(K(X)^{(n)})/C(K(A)^{(n)}) \to CK(X^{(n)}, A^{(n)})$.

We get a commutative diagram of chain maps where rows are exact:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & C(K(A)^{(n)}) & \longrightarrow & C(K(X)^{(n)}) & \longrightarrow & C(K(X)^{(n)})/C(K(A)^{(n)}) & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \theta(A)_\#} & & \downarrow{\scriptstyle \theta(X)_\#} & & \downarrow{\scriptstyle \theta(X,\ A)_\#} & & \\ 0 & \longrightarrow & CK(A^{(n)}) & \longrightarrow & CK(X^{(n)}) & \longrightarrow & CK(X^{(n)},\ A^{(n)}) & \longrightarrow & 0 \end{array} \tag{3.10}$$

(3.10) THEOREM. If A and X are CW complexex with $A \subseteq X$, then $\theta(X, A)_\#$ of (3.9) induces a natural isomorphism

$$H(K(X)^{(n)},\ K(A)^{(n)}) \to H(X^{(n)},\ A^{(n)}),$$

and θ induces a natural isomorphism of the exact homology sequence of $(K(X)^{(n)}, K(A)^{(n)})$ with the exact homology sequence of $(K(X^{(n)}), K(A^{(n)}))$.

PROOF. Consider the commutative diagram

$$\begin{array}{ccc} H(K(X)^{(n)}) & \xrightarrow{\ \theta(X)_*\ } & H(K(X^{(n)})) \\ {\scriptstyle \psi(K(X)^{(n)})}\downarrow & & \downarrow{\scriptstyle \psi(K(X^{(n)}))} \\ H(|K(X)^{(n)}|) & \xrightarrow{\ |\theta(X)|_*\ } & H(|K(X^{(n)})|) \end{array}$$

where ψ is the natural isomorphism of the homology of a simplicial set with the homology of its geometric realization (see [4], p. 62, 63). By (3.7), $|\theta(X)|$ is a weak homotopy equivalence. Hence $|\theta(X)|_*$ is an isomorphism (see [6], VII 25, p. 406). It follows that $\theta(X)_*$ is an isomorphism. Similarly, $\theta(A)_*$ is an isomorphism. By applying the five lemma to the long exact sequences of $(K(X)^{(n)}, K(A)^{(n)})$ and $(K(X^{(n)}), K(A^{(n)}))$ obtained from (3.10), $\theta(X, A)_*$ is proven to be an isomorphism also, and θ_* gives a natural isomorphism of these exact homology sequences. □

The foregoing theorem enables us to replace $H(X^{(n)}, A^{(n)})$ by $H(K(X)^{(n)}, K(A)^{(n)})$, whenever $A \subseteq X$ are CW complexes, in order to define the trace. First we define the <u>trace chain map</u> ν^n: $C(K(X)^{(n)}) \to CK(X)$ by the rule:

(3.11) $$\nu^n[\sigma_1, \ldots, \sigma_n] = \sigma_1 + \ldots + \sigma_n$$

for $\sigma_1, \ldots, \sigma_n \in K_m(X)$. Clearly ν^n is a well-defined chain map and ν^n maps $C(K(A)^{(n)})$ into $CK(A)$, and hence we have an induced chain map

$$\nu^n: \quad C(K(X)^{(n)})/C(K(A)^{(n)}) \to CK(X)/CK(A).$$

Passing to homology, ν^n induces a natural homomorphism from the exact homology sequence of $(K(X)^{(n)}, K(A)^{(n)})$ to that of (X, A).

(3.12) For CW complexes A and X with $A \subseteq X$ and for all $n > 0$, we define a <u>trace homology homomorphism</u>

$$\mu^n = \mu^n(X, A): \quad H(X^{(n)}, A^{(n)}) \to H(X, A)$$

to be the composition $\nu^n{}_*\theta_*^{-1}$

$$H(X^{(n)}, A^{(n)}) \xrightarrow{\theta_*^{-1}} H(K(X)^{(n)}, K(A)^{(n)}) \xrightarrow{\nu^n{}_*} H(X, A).$$

From (3.10) and (3.11), it follows that μ^n yields a natural homomorphism from the exact homology sequence of $(X^{(n)}, A^{(n)})$ to that of (X, A). In particular, for $g\colon (X, A) \to (Y, B)$ where X, A, Y, B are CW complexes, then $\mu^n(Y, B)g_*^{(n)} = g_*\mu^n(X, A)$.

The behavior of the trace relative to the diagonal map of (1.7) is given in the next lemma.

(3.13) LEMMA. For CW complexes $A \subseteq X$ and integers n, m and q with $n \cdot m = q$, the following diagram is commutative

$$\begin{array}{ccc} H(X^{(n)}, A^{(n)}) & \xrightarrow{d_*} & H(X^{(q)}, A^{(q)}) \\ \downarrow{\scriptstyle \mu^n} & & \downarrow{\scriptstyle \mu^q} \\ H(X, A) & \xrightarrow{m \cdot id} & H(X, A) \end{array}$$

where $d = d_n{}^q(X, A)$ is the diagonal map defined in (1.7).

PROOF: Define $\bar{d}\colon (K(X)^{(n)}, K(A)^{(n)}) \to (K(X)^{(q)}, K(A)^{(q)})$ by $\bar{d}[\sigma_1, \ldots, \sigma_n] = [\underbrace{\sigma_1, \ldots, \sigma_1}_{m}, \ldots, \underbrace{\sigma_n, \ldots, \sigma_n}_{m}]$, (that is, $\bar{d}$ repeats each simplex m times). Then $\bar{d}$ is simplicial and the following diagrams are easily seen to be commutative:

$$(3.14)\qquad \begin{array}{ccc} (K(X)^{(n)},\ K(A)^{(n)}) & \xrightarrow{\ \bar{d}\ } & (K(X)^{(q)},\ K(A)^{(q)}) \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\ (K(X^{(n)}),\ K(A^{(n)})) & \xrightarrow{\ d\ } & (K(X^{(q)},\ K(A^{(q)})), \end{array}$$

$$(3.15)\qquad \begin{array}{ccc} C(K(X)^{(n)})/C(K(A)^{(n)}) & \xrightarrow{\ \bar{d}\ } & C(K(X)^{(q)})/C(K(A)^{(q)}) \\ \downarrow{\scriptstyle\nu^n} & & \downarrow{\scriptstyle\nu^q} \\ C(K(X)/CK(A) & \xrightarrow{\ m\cdot id\ } & CK(X)/CK(A) \end{array}$$

Hence in diagram (3.16) below,

$$\begin{aligned} \mu^q d_* &= \nu_*{}^q \theta_*^{-1} d_* && \text{by } (3.12) \\ &= \nu_*{}^q \bar{d}_* \theta_*^{-1} && \text{by } (3.14) \\ &= m\nu_*{}^n \theta^{-1} && \text{by } (3.15) \\ &= m\mu^n && \text{by } (3.12). \end{aligned}$$

$$(3.16)\qquad \begin{array}{ccccc} H(K(X)^{(n)},\ K(A)^{(n)}) & & \xrightarrow{\ \bar{d}_*\ } & & H(K(X)^{(q)},\ K(A)^{(q)}) \\ \downarrow{\scriptstyle\nu_*{}^n} & \searrow{\scriptstyle\theta_*} & & \swarrow{\scriptstyle\theta_*} & \downarrow{\scriptstyle\nu_*{}^q} \\ & H(X^{(n)},\ A^{(n)}) & \xrightarrow{\ d_*\ } & H(X^{(q)},\ A^{(q)}) & \\ & \swarrow{\scriptstyle\mu^n} & & \searrow{\scriptstyle\mu^q} & \\ H(X,\ A) & & \xrightarrow{\ m\cdot id\ } & & H(X,\ A) \end{array}$$

□

Note that, for any spaces E and F, $K(E \times F)$ is naturally isomorphic to $K(E) \times K(F)$ by the correspondence $\sigma \longmapsto (p_1\sigma,\ p_2\sigma)$ where $\sigma \in K_m(E \times F)$ and $p_1\colon E \times F \to E$, $p_2\colon E \times F \to F$ are the projections. Henceforth, these simplicial sets will be identified.

For any two simplicial sets K and L, there is a natural chain homotopy equivalence

$$\text{(3.17)} \qquad EZ\colon\ C(K) \otimes C(L) \to C(K \times L)$$

(see [4], 29.6, p. 132) called an Eilenberg-Zilber map. Thus for CW complexes X, Y and positive integers n, m, naturality yields a commutative diagram of chain maps

$$\text{(3.18)} \qquad \begin{array}{ccc} CK(X^{(n)}) \otimes CK(Y^{(m)}) & \xrightarrow{EZ} & CK(X^{(n)} \times Y^{(m)}) \\ \uparrow{\scriptstyle \theta^n \otimes \theta^m} & & \uparrow{\scriptstyle \theta^n \times \theta^m} \\ C(K(X)^{(n)}) \otimes C(K(Y)^{(m)}) & \xrightarrow{EZ} & C(K(X)^{(n)} \times K(Y)^{(m)}) \end{array}$$

where θ here denotes the chain map of the simplicial map θ defined in (3.6).

(3.19) LEMMA. For all CW complexes X, Y and positive integers n, m,

$$(\theta^n \times \theta^m)_*\colon\ H(K(X)^{(n)} \times K(Y)^{(m)}) \to H(X^{(n)} \times Y^{(m)})$$

is an isomorphism.

PROOF. Let $H(K)^+$ denote the suspended homology of simplicial set K, i.e., $(H(K)^+)_p = H_{p-1}(K)$, for all p. Let $A*B$ denote the torsion product of groups A and B. We use the Künneth Theorem (see [1], 9.13, p. 64) for the free chain complexes $CK(X^{(n)})$, $CK(Y^{(m)})$ and $C(K(X)^{(n)})$, $C(K(Y)^{(m)})$ to get the exact columns of the commutative diagram:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
H(C(K(X)^{(n)}))\otimes H(C(K(Y)^{(m)})) & \xrightarrow{\theta_*{}^n \otimes \theta_*{}^m} & H(X^{(n)})\otimes H(Y^{(m)}) \\
\downarrow & & \downarrow \\
H(C(K(X)^{(n)})\otimes C(K(Y)^{(m)})) & \xrightarrow{(\theta^n \otimes \theta^m)_*} & H(CK(X^{(n)})\otimes CK(Y^{(m)})) \\
\downarrow & & \downarrow \\
H(C(K(X)^{(n)}))^+ * H(C(K(Y)^{(m)})) & \xrightarrow{\theta_*{}^n * \theta_*{}^m} & H(X^{(n)})^+ * H(Y^{(m)})^+ \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

Since $\theta_*{}^n$ and $\theta_*{}^m$ are isomorphisms by (3.10), the bottom and top horizontal arrows are isomorphisms. By the five lemma, $(\theta^n \otimes \theta^m)_*$ is also an isomorphism. Using (3.18) and passing to homology we get

$$(EZ)_*(\theta^n \otimes \theta^m)_* = (\theta^n \times \theta^m)_*(EZ)_*,$$

from which it follows that $(\theta^n \times \theta^m)_*$ is an isomorphism. □

(3.20) COROLLARY. <u>If $A \subseteq X$ and $B \subseteq Y$ are CW complexes, and n, m are positive integers, then</u>

$$(\theta^n \times \theta^m)_*: \ H(K(X)^{(n)} \times K(Y)^{(m)}, K(A)^{(n)} \times K(B)^{(m)})$$

$$\to \ H(X^{(n)} \times Y^{(m)}, A^{(n)} \times B^{(m)})$$

is an isomorphism.

PROOF. Apply the five lemma to the long exact sequences which arise from the commutative diagram of short exact sequences:

$$\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
C(K(A)^{(n)} \times K(B)^{(m)}) & \xrightarrow{\theta^n\times\theta^m} & CK(A^{(n)} \times B^{(m)}) \\
\downarrow & & \downarrow \\
C(K(X)^{(n)} \times K(Y)^{(m)}) & \xrightarrow{\theta^n\times\theta^m} & CK(X^{(n)} \times Y^{(m)}) \\
\downarrow & & \downarrow \\
C(K(X)^{(n)}\times K(Y)^{(m)})/C(K(A)^{(n)}\times K(B)^{(m)}) & \xrightarrow{\theta^n\times\theta^m} & CK(X^{(n)}\times Y^{(m)})/CK(A^{(n)}\times B^{(m)}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

Since the top two of the horizontal arrows induce homology isomorphisms (by 3.19) the bottom map does also. □

The relationship of the trace μ and adjoining map α of (1.4) will now be established.

(3.21) PROPOSITION. Let $A \subseteq X$ be CW complexes and n and m positive integers. Let p_1: $(X^{(n)} \times X^{(m)}, A^{(n)} \times A^{(m)}) \to (X^{(n)}, A^{(n)})$

and p_2: $(X^{(n)} \times X^{(m)}, A^{(n)} \times A^{(m)}) \to (X^{(m)}, A^{(m)})$ *be the coordinate projections. Then*

$$\mu^{n+m}\alpha_n{}^m{}_* = \mu^n p_{1*} + \mu^m p_{2*} \tag{3.22}$$

in the diagram

$$\begin{array}{ccccc}
H(X^{(n)} \times X^{(m)}, A^{(n)} \times A^{(m)}) & & \xrightarrow{p_{1*}} & & H(X^{(n)}, A^{(n)}) \\
\Big\downarrow p_{2*} & \searrow^{\alpha_n{}^m{}_*} & & & \Big\downarrow \mu^n \\
 & & H(X^{(n+m)}, A^{(n+m)}) & \searrow^{\mu^{n+m}} & \\
H(X^{(m)}, A^{(m)}) & & \xrightarrow{\mu^m} & & H(X, A) \; .
\end{array} \tag{3.23}$$

PROOF. Clearly the following diagram commutes.

$$\begin{array}{ccc}
(K(X)^{(n)}, K(A)^{(n)}) & \xrightarrow{\theta} & (K(X^{(n)}), K(A^{(n)})) \\
\Big\uparrow p_1 & & \Big\uparrow p_1 \\
(K(X)^{(n)} \times K(X)^{(m)}, K(A)^{(n)} \times K(A)^{(m)}) & \xrightarrow{\theta\times\theta} & (K(X^{(n)}) \times K(X^{(m)}), K(A^{(n)}) \times K(A^{(m)})) \\
\Big\downarrow p_2 & & \Big\downarrow p_2 \\
(K(X)^{(m)}, K(A)^{(m)}) & \xrightarrow{\theta} & (K(X^{(m)}), K(A^{(m)}))
\end{array} \tag{3.24}$$

Also, there is a commutative diagram

$$\begin{array}{ccc}
(K(X)^{(n)} \times K(X)^{(m)}, K(A)^{(n)} \times K(A)^{(m)}) & \xrightarrow{\theta\times\theta} & (K(X^{(n)} \times X^{(m)}), K(A^{(n)} \times A^{(m)})) \\
\Big\downarrow \bar{\alpha} & & \Big\downarrow \alpha \\
(K(X)^{(n+m)}, K(A)^{(n+m)}) & \xrightarrow{\theta} & (K(X^{(n+m)}), K(A^{(n+m)}))
\end{array} \tag{3.25}$$

where $\bar{\alpha}$ is the simplicial function given by

$\bar{\alpha}([\sigma_1, \ldots, \sigma_n], [t_1, \ldots, t_n]) = [\sigma_1, \ldots, \sigma_n, t_1, \ldots, t_m]$ for $\sigma_1, \ldots, \sigma_n, t_1, \ldots, t_m \in K_p(X)$, $p \geq 0$.

Now consider the chain map diagram where ν is the trace chain map defined in (3.11):

$$
\text{(3.26)} \qquad
\begin{array}{ccccc}
C(K(X)^{(n)} \times K(X)^{(m)}, K(A)^{(n)} \times K(A)^{(m)}) & & \xrightarrow{p_1} & & C(K(X)^{(n)}, K(A)^{(n)}) \\
\Big\downarrow p_2 & \searrow \bar{\alpha} & & & \Big\downarrow \nu^n \\
 & & C(K(X)^{(n+m)}, K(A)^{(n+m)}) & & \\
 & & & \searrow \nu^{n+m} & \\
C(K(X)^{(m)}, K(A)^{(m)}) & & \xrightarrow{\nu^m} & & C(K(X), K(A))
\end{array}
$$

One verifies easily that, in (3.26),

$$\nu^n p_1 + \nu^m p_2 = \nu^{n+m} \bar{\alpha}.$$

Passing to homology and first applying the isomorphism $(\theta \times \theta)_*^{-1}$, (see 3.20), we get

$$\nu_*^n \, p_{1*}(\theta \times \theta)_*^{-1} + \nu_*^m p_{2*}(\theta \times \theta)_*^{-1} =$$

$$\nu_*^{n+m} \bar{\alpha}_*(\theta \times \theta)_*^{-1}.$$

Using (3.24) and (3.25), we get

$$\nu_*^n \theta_*^{-1} p_{1*} + \nu_*^m \theta_*^{-1} p_{2*} = \nu^{n+m} \theta_*^{-1} \alpha_*$$

which by definition of trace (3.10) yields

$$\mu^n p_{1*} + \mu^m p_{2*} = \mu^{n+m} \alpha_*$$

as mappings of $H(X^{(n)} \times X^{(m)}, A^{(n)} \times A^{(m)})$ into $H(X, A)$. This establishes equation (3.22). □

Let × denote the homology exterior product.

(3.27) THEOREM. For CW complexes, $A \subseteq X$ and $B \subseteq Y$ and positive integers n and m, the following diagram commutes

$$\begin{array}{ccc}
H(X^{(n)},A^{(n)}) \otimes H(Y^{(m)},B^{(m)}) & \xrightarrow{\mu^n \otimes \mu^m} & H(X,A) \otimes H(Y,B) \\
\downarrow \times & & \\
H((X^{(n)},A^{(n)}) \times (Y^{(m)},B^{(m)}) & & \downarrow \times \\
\downarrow \pi_* & & \\
H((X \times Y)^{(nm)},(A \times Y \cup X \times B)^{(nm)}) & \xrightarrow{\mu^{nm}} & H((X,A) \times (Y,B))
\end{array}$$

where π is the product map of (1.10) and μ is the trace homomorphism.

PROOF. We use the MacLane-Map ∇ (see [2], p. 119, 5.8) as a specific choice for the Eilenberg-Zilber chain map and verify that the following diagram commutes:

$$\begin{array}{ccc}
C(K(X)^{(n)},K(A)^{(n)}) \otimes C(K(Y)^{(m)},K(B)^{(m)}) & \xrightarrow{\nu^n \otimes \nu^m} & CK(X,A) \otimes CK(Y,B) \\
\downarrow \nabla & & \\
C((K(X)^{(n)},K(A)^{(n)}) \times (K(Y)^{(m)},K(A)^{(m)})) & & \downarrow \nabla \\
\downarrow \pi & & \\
C(K(X \times Y)^{(nm)},K(A \times Y \cup X \times B)^{(nm)}) & \xrightarrow{\nu^{(nm)}} & CK((X,A) \times (Y,B)).
\end{array} \tag{3.28}$$

ν denotes the map of (3.11); π denotes the chain map given by

$$\pi([\sigma_1, \ldots, \sigma_n], [t_1, \ldots, t_m]) = [(\sigma_i, t_j): 1 \leq i \leq n, 1 \leq j \leq m]$$

for $[\sigma_1, \ldots, \sigma_n] \in K_j(X)^{(n)}$ and $[t_1, \ldots, t_m] \in K_j(Y)^{(n)}$, $j \geq 0$;

and $\nabla(x \otimes y) = \sum_{(\phi,\rho)} \varepsilon(\phi, \rho)(s_{\rho_r} \ldots s_{\rho_1} x, s_{\phi_p} \ldots s_{\phi_1} y)$

for dim x = p, dim y = r, (ϕ, ρ) runs through all (p, r)-shuffles, and

$$\varepsilon(\phi, \rho) = (-1)^{\eta} \quad \text{with} \quad \eta = \textstyle\sum_{i=1}^{p} \phi_i - i + 1.$$

Take $[\sigma_1, \ldots, \sigma_n] \in K_p((X)^{(n)})$, $[t_1, \ldots, t_n] \in K_r((Y)^{(n)})$. (In verifying commutativity of (3.28) we abbreviate the notation so that $\varepsilon = \varepsilon(\phi, \rho)$, $s = s_{\rho_n} \ldots s_{\rho_1}$, $s' = s_{\phi_p} \ldots s_{\phi_1}$ for any (p, r) shuffle (ϕ, ρ).) We have

$$\pi\nabla[\sigma_1, \ldots, \sigma_n] \otimes [t_1, \ldots, t_m] =$$

$$\nu\pi \sum_{(\phi,\rho)} \varepsilon\ (s[\sigma_1, \ldots, \sigma_n], s'[t_1, \ldots, t_m])$$

$$= \nu\pi \sum_{(\phi,\rho)} \varepsilon\ ([s\sigma_1, \ldots, s\sigma_n], [s't_1, \ldots, s't_m])$$

$$= \nu \sum_{(\phi,\rho)} \varepsilon\ [(s\sigma_i, s't_j): 1 \leq i \leq n, 1 \leq j \leq m]$$

$$= \sum_{(\phi,\rho)} \varepsilon \textstyle\sum_{i=1}^{n} \sum_{j=1}^{m} (s\sigma_i, s't_j)$$

$$= \textstyle\sum_{i=1}^{n} \sum_{j=1}^{m} \displaystyle\sum_{(\phi,\rho)} \varepsilon\ (s\sigma_i, s't_j)$$

$$= \textstyle\sum_{i=1}^{n} \sum_{j=1}^{m} \nabla(\sigma_i \otimes t_j)$$

$$= \nabla((\textstyle\sum_{i=1}^{n} \sigma_i) \otimes (\sum_{j=1}^{m} t_j))$$

$$= \nabla(\nu \otimes \nu)\ [\sigma_1, \ldots, \sigma_n] \otimes [t_1, \ldots, t_m].$$

Hence (3.28) commutes.

Note that $(\mu \otimes \mu)(\theta_* \otimes \theta_*) = \nu_* \otimes \nu_*$ by definition and since θ_* is an isomorphism

$$(\mu \otimes \mu) = (\nu_* \otimes \nu_*)(\theta_* \otimes \theta_*)^{-1}.$$

It follows that

$$(3.29) \qquad (\nu \otimes \nu)_*(\theta \otimes \theta)_*^{-1}\beta = \beta(\mu \otimes \mu)$$

where β denotes the canonical maps

β: $H(X, A) \otimes H(Y, B) \rightarrow H(CK(X, A) \otimes CK(Y, B))$ and

β: $H(X^{(n)}, A^{(n)}) \otimes H(Y^{(m)}, B^{(m)}) \rightarrow H(CK(X^{(n)}, A^{(n)}) \otimes CK(Y^{(m)}, B^{(m)}))$.

Consider the diagram:

$$\begin{array}{ccc}
H(X^{(n)}, A^{(n)}) \otimes H(Y^{(m)}, B^{(m)}) & & \\
\downarrow \beta & & \\
H(CK(X^{(n)},A^{(n)}) \otimes CK(Y^{(m)},B^{(m)})) & \xrightarrow{\pi_*\nabla_*} & H((X\times Y)^{(nm)}, (A\times Y \cup X\times B)^{(nm)}) \\
\uparrow (\theta \otimes \theta)_* & & \uparrow \theta_* \\
H(C(K(X)^{(n)},K(A)^{(n)}) \otimes C(K(Y)^{(m)},K(B)^{(m)})) & \xrightarrow{\pi_*\nabla_*} & H(K(X\times Y)^{(nm)}, K(A\times Y \cup X\times B)^{(nm)}) \\
\downarrow (\nu \otimes \nu)_* & & \downarrow \nu_* \\
H(CK(X, A) \otimes CK(Y, B)) & \xrightarrow{\nabla_*} & H((X, A) \times (Y, B)) \\
\uparrow \beta & & \\
H(X, A) \otimes H(Y, B) & &
\end{array}$$

The exterior product $\times$ is by definition $\nabla_*\beta$. By naturality of θ, the top rectangle commutes. By (3.10) and (3.19) θ_* and $(\theta \otimes \theta)_*$ are isomorphisms. By (3.28) the bottom rectangle commutes.

Take $x \in H\ (X^{(n)}, A^{(n)})$ and $y \in H(Y^{(m)}, B^{(m)})$

$$\begin{aligned} \mu\pi_*\times(x \otimes y) &= \nu_*\theta_*^{-1}\pi_*\nabla_*\beta(x \otimes y) \\ &= \nabla_*(\nu \otimes \nu)_*(\theta \otimes \theta)_*^{-1}\beta(x \otimes y) \\ &= \nabla_*\beta(\mu \otimes \mu)(x \otimes y) \quad \text{by (3.29)} \\ &= \times(\mu \otimes \mu)(x \otimes y). \end{aligned}$$

Hence $\mu\pi_*(x \times y) = (\mu x) \times (\mu y)$. This proves lemma (3.27). □

§4. The fixed-point index. Homotopy invariance.

(4.1) For each positive integer q, select a generator ω_q for $H_q(S^q)$ where S^q is the q-sphere. Let us identify S^q with $\mathbb{R}^q \cup \{\infty\}$ and let 0 denote the origin of $\mathbb{R}^q$. The inclusion $j: S^q \to (S^q, S^q - 0)$ induces an isomorphism on reduced homology since $S^q - 0$ is contractible, and the inclusion $k: (\mathbb{R}^q, \mathbb{R}^q - 0) \to (S^q, S^q - 0)$ is an excision. Thus $k_*^{-1}j_*: H_q(S^q) \to H_q(\mathbb{R}^q, \mathbb{R}^q - 0)$ is an isomorphism. Let $o_q = o_q(\mathbb{R}^q, 0)$ denote the image of ω_q under $k_*^{-1}j_*$. The generator o_q of the infinite cyclic group $H_q(\mathbb{R}^q, \mathbb{R}^q - 0)$ is called the fundamental class associated with $(\mathbb{R}^q, 0)$.

(4.3) Whenever K is a compact subset of an open subset V of R^q, we have inclusions

$$S^q \xrightarrow{\ i\ } (S^q, S^q - K) \xleftarrow{\ j\ } (V, V - K)$$

where j is an excision. We let $o_q(V, K)$ denote the image of o_q under the homomorphism $j_*^{-1}i_*$: $H_q(S^q) \to H_q(V, V - K)$. o_q is the fundamental class associated with (V, K).

(4.4) For an admissible map, the difference map (i - f), see (2.3), maps $(V, V - K_f)$ into $(E^{(n)}, (E - 0^{(n)})$ where $E = R^q$, V open in E, and K_f is the fixed point set of f. Consider the composition

$$H_q(V, V-K_f) \xrightarrow{(i-f)_*} H_q(E^{(n)}, (E-0)^{(n)}) \xrightarrow{\mu^n(E,E-0)} H_q(E,E-0),$$

where μ is the trace defined in (3.12). (Note that E - 0 and E are CW complexes.) The image of the fundamental class $o_q(V,K_f)$ under this homorphism is an integer multiple of o_q, say ao_q for some $a \in Z$. Define the fixed-point-index I(f) = a. The index is independent of the choice of orientation ω_q for S^q. I(f) is an integer, and in case n = 1, I(f) agrees with the classical index for single-valued maps as given by Dold (see [1], VII, 5.2).

(4.5) LEMMA. If the admissible map f: $V \to E^{(n)}$ with $E = \mathbb{R}^q$ has no fixed points, then I(f) = 0.

PROOF. If $K_f = \phi$, then $H^q(V, V - K_f) = 0$. □

(4.6) LEMMA. Let f: $V \to E^{(n)}$ be admissible with $E = \mathbb{R}^q$ and fixed point set K_f. If there are sets K and W with K compact, W open in V, and $K_f \subset K \subset W \subset V$, then the composition

$$H_q(W,W-K) \xrightarrow{(i-f)_*} H_q(E^{(n)}, (E - 0)^{(n)}) \xrightarrow{\mu} H_q(E, E - 0)$$

maps $o_q(W, K)$ into $I(f)o_q$.

This lemma asserts, in essense, that in computing $I(f)$, K_f may be replaced by a larger compact set and V replaced by a smaller open set.

PROOF. Consider the commutative diagram where unmarked arrows are induced by inclusions.

$$\begin{array}{ccccccc}
 & & H_q(S^q, S^q - K) & \xleftarrow{\cong} & H_q(W, W - K) & & \\
 & \nearrow & \downarrow & & \downarrow & \searrow (i-f)_* & \\
H_q(S^q) & & & & & & H_q(E^{(n)}, (E-O)^{(n)}) \xrightarrow{\mu} H_q(E, E-O) \\
 & \searrow & \downarrow & & \downarrow & \nearrow (i-f)_* & \\
 & & H_q(S^q, S^q - K_f) & \xleftarrow{\cong} & H_q(V, V - K_f) & &
\end{array}$$

By commutativity, $o_q(W, K) \longmapsto o_q(V, K_f)$ and also $\mu(i - f)_* o_q(W, K) = \mu(i - f)_* o_q(V, K_f) = I(f) o_q$. □

(4.7) HOMOTOPY INVARIANCE THEOREM. Let f be a homotopy from f_0 to f_1: $V \to E^{(n)}$, where V is open in $E = \mathbb{R}^q$, such that $K = \cup_{t \in I} K_{f_t}$ is compact. Then $I(f_0) = I(f_1)$.

PROOF. Define $G: V \times I \to E^{(n)}$ by $G(x, t) = (i - f_t)(x)$ for $(x, t) \in V \times I$, (see 2.3). G is the composition

$$V \times I \xrightarrow{\Delta \times id} V \times V \times I \xrightarrow{i \times f} E \times E^{(n)} \xrightarrow{D} E^{(n)}$$

where Δ is the diagonal, $i: V \subseteq E$, and D is defined in (2.2). Then G is a homotopy from $i - f_0$ to $i - f_1$ as maps of $(V, V - K)$ into $(E^{(n)}, (E - O)^{(n)})$. From the commutativity of the diagram

$$\begin{array}{ccc}
H_q(V,\ V - K) & \longrightarrow & H_q(V,\ V - K_{f_1}) \\
\downarrow & \searrow^{(i - f_0)_* = (i - f_1)_*} & \downarrow (i - f_1)_* \\
H_q(V,\ V - K_{f_0}) & \xrightarrow{(i - f_0)_*} & H_q(E^{(n)},\ (E - 0)^{(n)}),
\end{array}$$

$$I(f_0)o_q = \mu(i - f_0)_* o_q(V,\ K) = \mu(i - f_1)_* o_q(V,\ K) = I(f_1)o_q. \quad \square$$

(4.8) DIAGONAL PROPERTY. If $f\colon\ V \to E^{(n)}$ is admissible, with $E = \mathbb{R}^q$, and $d\colon\ E^{(n)} \to E^{(mn)}$ is an n-fold diagonal, $d = d_n^{mn}$ as in (1.7), then $K_{df} = K_f$, df is admissible, and $I(df) = mI(f)$.

PROOF. It is immediate from the definition of diagonal map that $K_{df} = K_f$. Since f is admissible, K_{df} is compact and df is admissible.

One easily checks that

$$d(i - f) = (i - df)\colon (V,\ V - K_f) \to (E^{(mn)},\ (E - 0)^{(mn)}).$$

Uisng (3.13), we get a commutative diagram

$$\begin{array}{ccc}
 & H_q(V,\ V - K_f) & \\
{}^{(i-f)_*}\swarrow & & \searrow^{(i-df)_*} \\
H_q(E^{(n)},(E - 0)^{(n)}) & \xrightarrow{\ d_*\ } & H_q(E^{(mn)},(E - 0)^{(mn)}) \\
\mu^n \downarrow & & \downarrow \mu^{mn} \\
H_q(E,\ E - 0) & \xrightarrow{\ m\cdot 1\ } & H_q(E,\ E - 0)
\end{array}$$

where 1 denotes the identity isomorphism of $H_q(E,\ E - 0)$. Hence

$$I(df)o_q = \mu^{mn}(i - df)_* o_q(V, K_f)$$

$$= m\mu^n(i - f)_* o_q(V, K_f)$$

$$= mI(f)o_q. \qquad \square$$

§5. <u>The Additivity Properties</u>. The first additivity property expresses the local nature of the fixed point index and is a direct generalization of the additivity rule for single valued functions.

(5.1). UNION ADDITIVITY PROPERTY. <u>Given $f\colon V \to E^{(n)}$ admissible, with $E = \mathbb{R}^q$, where $V = \bigcup_{i=1}^k V_i$ and each V_i is open in E. Assume that the fixed point set K_i of $f|V_i$ is compact for each i and that $K_i \cap K_j = \phi$ whenever $i \neq j$. Then</u>

$$I(f) = \sum_{i=1}^k I(f|V_i).$$

PROOF. Clearly $K_f = \bigcup_{i=1}^k K_i$. Since V is Hausdorff and $\{K_1, \ldots, K_k\}$ is a finite collection of compact subsets of V, we can find open sets $\{W_1, \ldots, W_k\}$ so that $K_i \subseteq W_i \subseteq V_i$ for each i and $W_i \cap W_j = \phi$ for $i \neq j$. Let $W = \bigcup_{i=1}^k W_i$. Then $f|W$ and $f|W_i$ are admissible functions and thus have a well defined index, for each i. Using (4.6), we obtain $I(f) = I(f|W)$ and $I(f|V_i) = I(f|W_i)$ for all i.

Using the direct sum representations induced by inclusions, we get a commutative diagram

$$\begin{array}{ccccc}
H_q(S^q) & \longrightarrow & H_q(S^q,\ S^q - K_f) & & \\
 & & \uparrow \cong & & \\
 & & H_q(W,\ W - K_f) & \xleftarrow{\ \cong\ } & \oplus_{j=1}^{k} H_q(W_j,\ W_j - K_j) \\
 & & \downarrow (i - f|W)_* & \swarrow \sum_{j=1}^{k}(i - f|W_j)_* & \\
 & & H_q(E^{(n)},(E-0)^{(n)}) & & \\
 & & \downarrow u^n(E,E-0) & & \\
 & & H_q(E,\ E - 0) & &
\end{array}$$

It follows that

$$I(f|W)o_q = \sum_{i=1}^{k} I(f|W_i)o_q,$$

and hence

$$I(f) = \sum_{i=1}^{k} I(f|V_i). \qquad \square$$

A second type of additivity applies to the adjunction operator of (1.9).

(5.2) ADJUNCTION ADDITIVITY PROPERTY. Given admissible maps $f_i\colon V \to E^{(n_i)}$, $i = 1, \ldots, k$, with $E = \mathbb{R}^q$, then $f_1 \vee \ldots \vee f_k\colon V \to E^{(n)}$ is admissible, where $n = n_1 + \ldots + n_k$, with $K_{f_1 \vee \ldots \vee f_k} = \cup_{i=1}^{k} K_{f_i}$, and $I(f_1 \vee \ldots \vee f_k) = I(f_1) + \ldots + I(f_k)$.

PROOF. Since adjunction is associative it is sufficient to verify the property in case $k = 2$; the general case will follow by induction.

One proves easily that $K_{f_1 \vee f_2} = K_{f_1} \cup K_{f_2}$. Since f_1 and f_2 are admissible, $K_{f_1 \vee f_2}$ is the union of two compact sets and is therefore compact. One checks also that

$$(5.3)\quad i - (f_1 \vee f_2) = (i - f_1) \vee (i - f_2) \colon (V,\ V - K_{f_1 \vee f_2}) \to (E^{(n)},\ (E - 0)^{(n)})$$

where $i\colon\ V \subseteq E$ and $n = n_1 + n_2$.

Now consider the (non-commutative) diagram in which p_1 and p_2 are the first and second projections of $E^{(n_1)} \times E^{(n_2)}$ on its coordinate spaces.

$$\begin{array}{ccccc}
 & & H_q(V,\ V - K_{f_1 \vee f_2}) & & \\
 & & \big\downarrow ((i-f_1)\times(i-f_2))_* & & \\
 & & H_q(E^{(n_1)} \times E^{(n_2)},\ (E-0)^{(n_1)} \times (E-0)^{(n_2)}) & & \\
 & \swarrow p_{1*} & \big\downarrow \alpha_* & \searrow p_{2*} & \\
H_q(E^{(n_1)}, (E-0)^{(n_1)}) & & H_q(E^{(n)},\ (E-0)^{(n)}) & & H_q(E^{(n_2)}, (E-0)^{(n_2)}) \\
 & \searrow \mu^{n_1} & \big\downarrow \mu^{n} & \swarrow \mu^{n_2} & \\
 & & H_q(E,\ E - 0) & &
\end{array}$$

Note that $p_1 \circ (i - f_1) \times (i - f_2) = i - f_1$ and $p_2 \circ (i - f_1) \times (i - f_2) = i - f_2$.

By definition of index (4.4) and adjunction (1.9), and by (5.3), we have

$$\begin{aligned}
I(f_1 \vee f_2)o_q &= \mu^n(i - f_1 \vee f_2)_* o_q(V,\ K_{f_1 \vee f_2}) \\
&= \mu^n((i - f_1) \vee (i - f_2))_* o_q(V,\ K_{f_1 \vee f_2})
\end{aligned}$$

$$= \mu^n \alpha_*((i - f_1) \times (i - f_2))_* o_q(V, K_{f_1 \vee f_2}).$$

But by (3.21), $\mu^{n_1} p_{1*} + \mu^{n_2} p_{2*} = \mu^n \alpha_*$. Hence $I(f_1 \vee f_2)o =$

$$[\mu^{n_1} p_{1*}((i - f_1) \times (i - f_2))_* + \mu^{n_2} p_{2*}((i - f_1) \times (i - f_2))_*] o_q(V, K_{f_1 \vee f_2})$$

$$= [\mu^{n_1}(i - f_1)_* + \mu^{n_2}(i - f_2)_*] o_q(V, K_{f_1 \vee f_2})$$

$$= [I(f_1) + I(f_2)] o_q.$$

Hence $I(f_1 \vee f_2) = I(f_1) + I(f_2)$. □

(5.4) COROLLARY. Let $f: V \to E^{(n)}$ be a constant map with constant value $[y_1, \ldots, y_n]$, V open in $E = \mathbb{R}^q$. Then f is admissible and $I(f)$ is the number of values of i for which $y_i \in V$.

PROOF. $K_f = \{y_1, \ldots, y_n\} \cap V$ is finite and hence compact. $f = f_1 \vee \ldots \vee f_n$ where $f_i: V \to E$ is the (single-valued) function with constant value y_i, $i = 1, \ldots, n$. By adjunction additivity $I(f) = \sum_{i=1}^n I(f_i)$, and $I(f_i) = 1$ or 0, depending on whether $y_i \in V$ or not. □

§6. The Multiplicative Property. If $E = \mathbb{R}^q$ and $F = \mathbb{R}^s$, we may identify $E \times F$ with $\mathbb{R}^{q+s}$. Then

$$((E - 0) \times F) \cup (E \times (F - 0)) = E \times F - 0, \quad \text{and}$$

$$(E, E - 0) \times (F, F - 0) = (E \times F, E \times F - 0).$$

It follows from the Künneth Theorem (see [1], VII 2.6) that exterior homology product in dimension $q + s$

$$H_q(E,\ E - O) \otimes H_s(F,\ F - O) \xrightarrow{\times} H_{q+s}(E \times F,\ E \times F - O)$$

is an isomorphism. Hence

$$(6.1) \qquad o_q \times o_s = \pm o_{q+s}.$$

We may choose the orientation so that the plus sign prevails. More generally, it can be proved (see [1], VII 2.15) that if K, K' are compact subsets of sets V, V', respectively, where V is open in E and V' is open in F, then

$$(6.2) \qquad o_q(V,\ K) \times o_s(V',\ K') = \pm o_{q+s}(V \times V',\ K \times K'),$$

and by choosing the orientations suitably the plus sign may be used.

(6.3) MULTIPLICATIVE PROPERTY. Let $f: V \to E^{(n)}$ and $g: W \to F^{(m)}$ be admissible with $E = \mathbb{R}^q$ and $F = \mathbb{R}^s$. Then $\pi(f \times g): V \to W$ $(E \times F)^{(nm)}$ is admissible with $K_{\pi(f\times g)} = K_f \times K_g$ and $I(\pi(f \times g)) = I(f) \cdot I(g)$.

PROOF. $\pi(f \times g)(x, y)$ has (x, y) as a coordinate if and only if x is a coordinate of $f(x)$ and y is a coordinate of $g(y)$. Hence $K_{\pi(f\times g)} = K_f \times K_g$, which is compact, and so $\pi(f \times g)$ is admissible

By definition of index (4.4) of $\pi(f \times g)$, and by (6.2), we have

$$I(\pi(f \times g))o_{q+s} = \mu^{nm}(i - \pi(f \times g))_* o_{q+s}(V \times W,\ K_f \times K_g)$$

$$= \mu^{nm}(i - \pi(f \times g))_* o_q(V,\ K_f) \times o_s(W,\ K_g),$$

where $i: V \times W \subseteq E \times F$.

One easily verifies that, if $i'\colon V \subseteq E$ and $i''\colon W \subseteq F$, then $i = i' \times i''$ and $(i - \pi(f \times g)) = \pi(i' - f) \times (i'' - g)$ as indicated in the commutative diagram of space pairs:

$$\begin{array}{ccc} & (V,\ V - K_f) \times (W,\ W - K_g) & \\ {\scriptstyle (i'-f)\times(i''-g)} \swarrow & & \searrow {\scriptstyle i-\pi(f\times g)} \\ (E^{(n)},\ (E\text{-}O)^{(n)})\times(F^{(m)},(F\text{-}O)^{(m)}) & \xrightarrow{\ \pi\ } & ((E\times F)^{(nm)},(E\times F\text{-}O)^{(nm)}). \end{array}$$

Hence $I(\pi(f \times g))o_{q+s} =$

$$\mu^{nm}\pi_*((i' - f) \times (i'' - g))_* o_q(V,\ K_f) \times o_s(W,\ K_g)$$

$$= \mu^{nm}\pi_*[(i' - f)_* o_q(V,\ K_f)] \times [(i'' - g)_* o_s(W,\ K_g)]$$

by naturality of the exterior homology product (see [1], VII, 2.7). Now (3.27) applies to give $I(\pi(f \times g)o_{q+s} =$

$$[\mu^n(i' - f)_* o_q(V,\ K_f)] \times [\mu^m(i'' - g)_* o_s(W,\ K_g)]$$

$$= [I(f)o_q] \times [I(g)o_s] = I(f)\cdot I(g)o_q \times o_s$$

$= I(f)\cdot I(g)o_{q+s}$, the last equality by (6.1). Hence

$I(\pi(f \times g)) = I(f)I(g).$ □

§7. <u>The Lefschetz Number</u>. We show in this section that the index of an admissible map $f\colon V \to E^{(n)}$ which factors $V \xrightarrow{h} P^{(n)} \hookrightarrow E^{(n)}$, where P is a compact CW complex $\subseteq V$, is equal to the Lefschetz number of the endomorphism

$$\mu(h|K)_*\colon H(P;\ Q) \to H(P^{(n)};\ Q) \to H(P,\ Q)$$

where Q is the field of rationals, and μ here denotes the homomorphism on rational homology induced by the trace homomorphism of integral homology.

We start with some algebraic preliminaries. All $\otimes$-products and Hom are over Q. Let $M = \{M_i\}_{i \in Q}$ and $N = \{N_i\}_{i \in \mathbb{Z}}$ be graded Q-modules, then Hom(M, N) denotes the graded Q-module where

$$(7.1) \qquad \mathrm{Hom}(M, N)_n = \Pi_{i \in \mathbb{Z}} \mathrm{Hom}_Q(M_i, N_{i+n}).$$

The <u>dual</u> module M* of M is the graded Q-module where

$$(7.2) \qquad (M^*)_i = \mathrm{Hom}_Q(M_{-i}, Q), \quad \text{for } i \in \mathbb{Z}.$$

A natural homomorphism (of degree 0) of graded Q-modules $\Theta = \Theta_{MN}$: $M^* \otimes N \to \mathrm{Hom}(M, N)$ is defined by the rule

$$(7.3) \qquad [\Theta(\phi \otimes n)](m) = (-1)^{|m||n|}\phi(m)n, \quad \text{if } |\phi| = -|m|$$
$$= 0 \qquad , \quad \text{if } |\phi| \neq -|m|.$$

It may be proven (see [1], p. 208, VII 6.3) that the image of Θ consists of those homomorphisms β: M → N of finite rank, and Θ is a nomomorphism (since Q is a field).

The <u>evaluation</u> homomorphism e: $N^* \otimes N \to Q$ is defined by

$$(7.4) \qquad e(\phi \otimes n) = \phi(n) \ , \quad \text{if } |\phi| = -|n|$$
$$= 0 \quad , \quad \text{if } |\phi| \neq -|n|,$$

for $\phi \in N^*$, $n \in N$.

(7.5) For an endomorphism β: N → N of finite rank, the <u>Lefschetz number</u> $\Lambda(\beta)$ is defined to be $e\Theta^{-1}(\beta) \in Q$.

It can be shown that $\Lambda(\beta)$ can be expressed in the more familiar form of alternating sum of traces of endomorphisms, (see [1], p. 208, VII, 6.4).

(7.6) THE LEFSCHETZ NUMBER PROPERTY. Given $f: V \to E^{(n)}$, with V open in $E = \mathbb{R}^q$, $q > 0$, $n > 0$, and given a compact CW complex $P \subseteq V$ for which f factors $f = i^{(n)}h$

$$V \xrightarrow{h} P^{(n)} \xrightarrow{i^{(n)}} E^{(n)}$$

with $i: P \subseteq E$. Then f is admissible, $\mu(h|P)_*: H(P; Q) \to H(P; Q)$ has finite rank, and the Lefschetz number $\Lambda(\mu(h|P)_*) = I(f)$.

PROOF. The fixed point set K_f is closed in V by (1.12) and contained in P. Hence K_f is compact and f is admissible. Furthermore, by lemma 4.6, $I(f)$ may be computed by replacing (in definition 4.4) the fixed-point set K_f by the larger compact set P. Thus $o_q(V, K) \longmapsto I(f)o_q$ via the composition in the top line of the diagram:

(7.7)
$$\begin{array}{ccccc} H_q(V,V\text{-}P;Z) & \xrightarrow{(i-f)_*} & H_q(E^{(n)},(E\text{-}O)^{(n)};Z) & \xrightarrow{\mu} & H_q(E,E\text{-}O;Z) \\ \downarrow & & \downarrow & & \downarrow \\ H_q(V,V\text{-}P;Q) & \xrightarrow{(i-f)_*} & H_q(E^{(n)},(E\text{-}O)^{(n)};Q) & \xrightarrow{\mu} & H_q(E,E\text{-}O;Q). \end{array}$$

The vertical arrows of (7.7) are induced by inclusion $Z \to Q$, and the diagram commutes. Let $o_q(V, K) \longmapsto \bar{o} \in H_q(V, V - P; Q)$ and $o_q \longmapsto \bar{\bar{o}} \in H_q(E, E - O; Q)$ via vertical arrows. Then $\bar{o} \longmapsto I(f)\bar{\bar{o}}$ by the composition of the bottom line of (7.7).

For the remainder of the proof, H(,) will denote homology with coefficients in the rationals Q. The Künneth formula gives a canonical isomorphism which we write as equality:

$$H((X, A) \times (Y, B)) = H(X, A) \otimes H(Y, B)$$

for all (X, A), (Y, B).

Let $D\colon (V, V - P) \times P^{(n)} \to (E^{(n)}, (E - O)^{(n)})$

and $D'\colon (V, V - P) \times P \to (E, E - O)$

be the difference maps defined in (2.3); that is,

$D'(x, y) = x - y$, $D(x, [y_1, \ldots, y_n]) = [x - y_1, \ldots, x - y_n]$, for $x \in V$, $y \in P$. Then one can show easily that $D_* = D'_*(1 \otimes \mu)$ in the diagram

$$(7.8)\qquad \begin{array}{ccc} H(E^{(n)}, (E - O)^{(n)}) & \xrightarrow{\ \mu\ } & H(E, E - O) \\ \uparrow{\scriptstyle D_*} & & \downarrow{\scriptstyle D'_*} \\ H(V, V - P) \otimes H(P^{(n)}) & \xrightarrow{\ 1\otimes\mu\ } & H(V, V - P) \otimes H(P) \end{array}$$

Now consider the following diagram whose top line sends $\bar{o}$ into $I(f)\bar{\bar{o}}$.

$$(7.9)\qquad \begin{array}{ccccc} H(V, V - P) & \xrightarrow{(i-f)_*} & H(E^{(n)}, (E-O)^{(n)}) & \xrightarrow{\ \mu\ } & H(E, E - O) \simeq Q \\ \downarrow{\scriptstyle \Delta_*} & & \uparrow{\scriptstyle D_*} & & \uparrow{\scriptstyle e} \\ H(V,V-P)\otimes H(V) & \xrightarrow{id\otimes h_*} & H(V,V-P)\otimes H(P^{(n)}) & \xrightarrow{\hat{D}\otimes\mu} & (HP)^* \otimes HP \end{array}$$

Here, $(HP)^*$ is the dual graded module of HP (see 7.2), e is the evaluation (see 7.4), Q may be identified with $H(E, E - O) = H_q(E, E - O)$ via $1 \leftrightarrow \bar{\bar{o}}$, and for any $x \in H(V, V - P)$ $\hat{D}(X)$: $HP \to Q$ is defined by

$$\hat{D}(x)(y) = D_*^!(x \otimes y) \quad \text{if } |x| + |y| = q$$
$$= 0 \qquad \text{otherwise,}$$

for $y \in HP$. The left square of (7.9) commutes by definition of D. In the right square,

$$e(\hat{D} \otimes \mu)(x \otimes y) = e(\hat{D}x \otimes \mu y) = \hat{D}(x)(\mu y)$$
$$= D_*^!(x \otimes \mu y) = \mu D_*(x \otimes y),$$

with the last equality obtained by (7.8). Hence (7.9) is commutative.

(7.10) Let $a = (\hat{D} \otimes \mu)(id \otimes h_*)\Delta_*\bar{o} \in (HP)^* \otimes HP$.

Let t: $V \times P \to P \times V$ be the map $t(x, y) = (y, x)$. Then on homology, t_*: $HV \otimes HP \to HP \otimes HV$ satisfies $t_*(x \otimes y) = (-1)^{|x||y|}(y \otimes x)$.

(7.11) Let $b = \Delta_*\bar{o} \in H(V, V - P) \otimes HV$.

Thus $(\hat{D} \otimes \mu h_*)(b) = a$.

The following diagram commutes:

$$
(7.12)\qquad
\begin{array}{ccccc}
HP & & & & \\
\downarrow{\scriptstyle (b\otimes -)} & & & & \\
H(V,V-P)\otimes HV\otimes HP & \xrightarrow{\hat{D}\otimes h_*\otimes id} & (HP)^*\otimes HP^{(n)}\otimes HP & & \\
\downarrow{\scriptstyle id\otimes t_*} & & \downarrow{\scriptstyle id\otimes t_*} & & \\
H(V,V-P)\otimes HP\otimes HV & \xrightarrow{\hat{D}\otimes id\otimes h_*} & (HP)^*\otimes HP\otimes HP^{(n)} & \xrightarrow{id\otimes id\otimes \mu} & (HP)^*\otimes HP\otimes HP \\
\downarrow{\scriptstyle D'_*\otimes id} & & \downarrow{\scriptstyle e\otimes id} & & \downarrow{\scriptstyle e\otimes id} \\
H(E,E-O)\otimes HV & \xrightarrow{id\otimes h_*} & H(E,E-O)\otimes HP^{(n)} & \xrightarrow{id\otimes \mu} & H(E,E-O)\otimes HP \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
HV & \xrightarrow{h_*} & HP^{(n)} & \xrightarrow{\mu} & HP
\end{array}
$$

The vertical isomorphisms at the bottom of (7.12) are given by $c \otimes z \longmapsto cz$ for $c \in H(E, E - O) = H_q(E, E - O) = Q$.

By tracing along the uppermost edges of diagram (7.12), one verifies that the composition $HP \to HP$ from top to bottom right in (7.12) is the homomorphism $\Theta(a)$.

However the composition of homomorphism along the left edge of diagram (7.12) is precisely $i_*: HP \to HV$ induced by inclusion $i: P \subseteq V$ (see [1], VII, 6.13, p. 210). By commutativity of (7.12), $\Theta(a)=\mu h_* i_*$. Hence, $\mu h_* i_*$ has finite rank and $a = \Theta^{-1}(\mu h_* i_*)$. By commutativity of (7.9), and by definition of index

$$\mu(i - f)_*\bar{o} = I(f)\bar{\bar{o}} = e(a)\bar{\bar{o}}.$$

Hence, $I(f) = e\Theta^{-1}(\mu h_* i_*) = \Lambda(\mu(h|P)_*)$. □

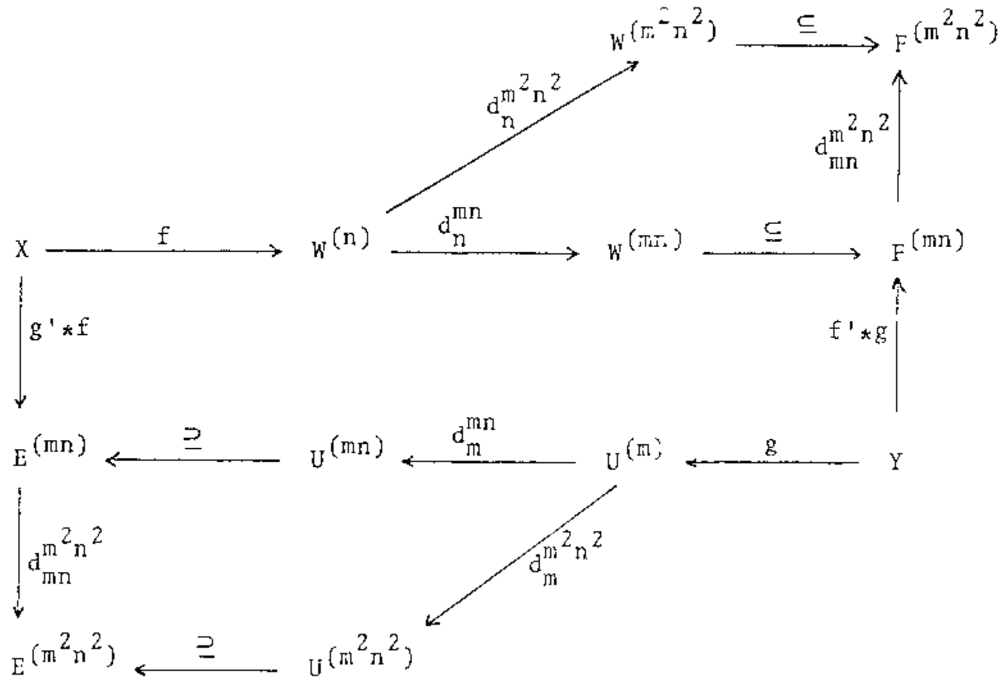
§8. The Commutativity Property. In this section, we study the fixed-point-index of composite maps as defined in (1.10). Under certain circumstances, the index will be invariant when the order of application of two symmetric product maps is reversed. First we present some preliminaries.

(8.1) Let $p: X^n \to X^{(n)}$ be the identification map and $p_1: X^n \to X$ be the projection on the first coordinate. If $G \subseteq X^{(n)}$, then $p_1p^{-1}(G) = \{y \in X \mid y$ is a coordinate of some element in $G\}$. If G is compact, then $p^{-1}(G)$ is compact (see [7], p. 160, Lemma 6.3) and hence $p_1p^{-1}(G)$ is compact. Thus p_1p^{-1} becomes a correspondence from subsets of $X^{(n)}$ to subsets of X which carries compact sets to compact sets.

(8.2) Let U be open in $E = \mathbb{R}^q$, $q > 0$, and W open in $F = \mathbb{R}^s$, $s > 0$. Let $f'\colon U \to F^{(n)}$ and $g'\colon W \to E^{(m)}$ be continuous where $n > 0$ and $m > 0$.

Put $X = f^{-1}W^{(n)}$ and $Y = g^{-1}\ U^{(m)}$. (We assume X and Y are non-empty). Then f' and g' defined maps $f\colon X \to W^{(n)}$ and $g\colon Y \to U^{(m)}$, respectively, by $f(x) = f'(x)$, $g(y) = g'(y)$.

The compositions $f'*g\colon Y \to F^{(nm)}$ and $g'*f\colon X \to E^{(nm)}$ are defined as in (1.8). The relevant functions may be displayed in the following diagram:



where the d maps are diagonals as defined in (1.7).

We define two homotopies $G, G' : X \times Y \times I \to (E \times F)^{(m^4 n^4)}$ by the rules:

$$\text{(8.4)} \quad \begin{aligned} G_t(x, y) &= \pi(A(tg'*fx, (1 - t)d_m^{mn}gy), d_n^{m^2n^2}fx) \\ G'_t(x, y) &= \pi(d_m^{m^2n^2}gy, A(tf'*gy, (1 - t)d_n^{mn}fx)) \end{aligned}$$

where $x \in X$, $y \in Y$, $0 \leq t \leq 1$, and π is the product map of (1.10) and A is the addition map of (2.4).

(8.5) COMMUTATIVITY THEOREM. Let f, g, G, and G' be as described in (8.2) and (8.4). If $\cup_{t\in I} K_{G_t}$ and $\cup_{t\in I} K_{G'_t}$ are compact, then f'*g and g'*f are admissible and I(f'*g) = I(g'*f).

PROOF. K_{G_0} is closed in $X \times Y$ by (1.12) and is a subset of the compact set $\cup_{t \in I} K_{G_t}$. Hence G_0 is admissible, and is given by

$$\begin{aligned} G_0(x, y) &= \pi(A(0, d_m^{mn} gy), d_n^{m^2n^2} fx) && \text{by (8.4)} \\ &= \pi(d_{mn}^{m^2m^2} d_m^{mn} gy, d_n^{m^2n^2} fx) && \text{by (2.5)} \\ &= \pi(d_m^{m^2n^2} gy, d_n^{m^2n^2} fx) \end{aligned}$$

Furthermore $K_{G_0} = K_{g'*f} \times K_{f'*g}$. For, (x, y) is a coordinate of $\pi(d_m^{m^2n^2} gy, d_n^{m^2n^2} fx)$ if and only if x is a coordinate of gy and y is a coordinate of fx. Hence, $K_{g'*f}$ and $K_{f'*g}$ are also compact, and $g'*f$ and $f'*g$ are admissible.

We will now show that $I(G_0) = m^3n^3 I(g'*f)$. The compactness condition of the homotopy invariance theorem (4.7) is satisfied, so we get $I(G_0) = I(G_1)$. Also,

$$(8.6) \qquad \begin{aligned} G_1(x, y) &= \pi(A(g'*fx, 0), d_n^{m^2n^2} fx) \\ &= \pi(d_{mn}^{m^2n^2} g'*fx, d_n^{m^2n^2} fx). \end{aligned}$$

Now choose $y_0 \in Y$ and define a homotopy $L: X \times Y \times I \to (E \times F)^{(m^4n^4)}$ by

$$(8.7) \quad L_t(x, y) = \pi(d_{mn}^{m^2n^2} g'*fx, A(td_1^{mn} y_0, (1 - t)d_n^{mn} fx)).$$

If (x, y) is a coordinate of $L_t(x, y)$ for some t, then $x \in K_{g'*f}$ and y is an element of the convex hull N of $\{y_0\} \cup p_1 p^{-1} f(K_{g'*f})$, (see (8.2)). N is the convex hull of a compact set and is therefore compact. Now $\cup_{t \in I} K_{L_t}$ is a subset of the compact set

$K_{g'*f} \times N$. By (1.13), $\cup_{t\in I} K_{L_t}$ is compact. Using homotopy invariance (4.7) again, $I(L_0) = I(L_1)$. From (8.6) and (8.7) we get

$$\begin{aligned} L_0(x, y) &= \pi(d_{mn}^{m^2n^2} g'*fx, A(0, d_n^{mn}fx)) \\ &= \pi(d_{mn}^{m^2n^2} g'*fx, d_n^{m^2n^2} fx) \\ &= G_1(x, y). \end{aligned}$$

$$\begin{aligned} L_1(x, y) &= \pi(d_{mn}^{m^2n^2} g'*fx, A(d_1^{mn}y_0, 0)) \\ &= \pi(d_{mn}^{m^2n^2} g'*fx, d_1^{m^2n^2} y_0). \end{aligned}$$

Hence $L_1 = \pi((d_{mn}^{m^2n^2} g'*f) \times (d_1^{m^2n^2} y_0))$, see (1.11). The constant map y_0: $Y \to E$ has index $I(y_0) = 1$ (see (5.4)). The diagonal property (4.8) and multiplicative property (6.3) yield

$$\begin{aligned} I(L_1) &= I(d_{mn}^{m^2n^2} g'*f)\cdot I(d_1^{m^2n^2} y_0) \\ &= mnI(g'*f)\cdot m^2n^2\cdot 1. \end{aligned}$$

Hence

$$(8.8) \qquad I(G_0) = I(G_1) = I(L_0) = I(L_1) = m^3n^3I(g'*f).$$

Similarly, we show that $I(G_0') = m^3n^2I(f'*g)$. We may use the homotopy invariance (4.7), to prove that $I(G_0') = I(G_1')$, and observe from (8.4) that

$$\begin{aligned} G_0'(x, y) &= \pi(d_m^{m^2n^2} gy, A(0, d_n^{mn}fx) \\ &= \pi(d_m^{m^2n^2} gy, d_n^{m^2n^2} fx), \quad \text{and} \end{aligned}$$

$$G_1'(x, y) = \pi(d_m^{m^2n^2} gy, A(f'*gy, 0))$$
$$= \pi(d_{mn}^{m^2n^2} gy, d_{mn}^{m^2n^2} f'*gy).$$

Choose $x_0 \in X$. Define a homotopy $L': X \times Y \times I \to (E \times F)^{(m^4n^4)}$ by

$$L_t'(x, y) = \pi(A(td_1^{mn}x_0, (1 - t)d_m^{mn}gy), d_{mn}^{m^2n^2} f'*gy).$$

One shows, in analogy with the corresponding proof for the homotopy L, that the compactness condition of (4.7) is satisfied for L'. Also

$$L_0' = G_1' \quad \text{and}$$
$$L_1' = \pi((d_1^{m^2n^2} x_0) \times d_{mn}^{m^2n^2} f'*gy)).$$

Hence

$$I(G_0') = I(L_0') = I(L_1') \tag{8.9}$$
$$= m^3n^3 I(f'*g),$$

the last equality by the multiplicative and diagonal properties. However, $G_0 = G_0'$. Therefore (8.8) and (8.9) combine to give the conclusion $I(f'*g) = I(g'*f)$. □

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