ALMOST PERIODIC ANALYTIC FUNCTIONS IN THE UNIT DISC

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ABSTRACT. We show that the only bounded analytic functions in the unit disc, that are almost periodic with respect to the group of Möbius transformations which have $z = 1$ as fixed point, are the constant functions. The main purpose of this article is to throw open the question of determining exactly which subgroups of the Möbius group allow non-constant bounded analytic functions that are almost periodic. Our main theorem may be regarded as a first step in this direction.

To be more precise, let $\mathbb{D} = \{|z| < 1\}$ be the unit disc in the complex plane $\mathbb{C}$, and let $H^\infty(\mathbb{D})$ be the space of all bounded analytic functions on $\mathbb{D}$, in the supremum norm topology. Let M be the Mobius group of all one:one conformal maps φ of $\mathbb{D}$ onto $\mathbb{D}$;

$$\varphi(z) = e^{i\lambda} \frac{z+z_o}{1+\bar{z}_o z}, \quad z_o \in \mathbb{D}, \quad \lambda \in \mathbb{R}.$$

Definition. If G is a subgroup of M and if $f \in H^\infty(\mathbb{D})$, then we say that f is G almost periodic if the orbit of f under G,

$$\mathrm{orb}(f:G) = \{f \circ \varphi : \varphi \in G\},$$

is a precompact subset of $H^\infty(\mathbb{D})$.

Definition. Let G_o be the subgroup of M consisting of all $\varphi \in M$ with $\varphi(1) = 1$. Let G_{oo} be the subgroup of M consisting of all $\varphi \in M$ with $\varphi(1) = 1$ and $\varphi(-1) = -1$.

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Note. G_o consists of all maps

$$\varphi(z) = \frac{1+\bar{z}_o}{1+z_o} \quad \frac{z+z_o}{1+\bar{z}_o z}, \qquad z_0 \in \mathbb{D}.$$

Thus we see that G_o is a group with two real parameters, and it is locally compact as a closed subgroup of M. We may map $\mathbb{D}$ conformally onto the upper half-plane $\mathbb{H} = \{\mathrm{Im}\, z > 0\}$ via $w = i(z+1)/(z-1)$ so that $z = -1$ goes to $w = 0$ and $z = +1$ goes to $w = \infty$. Thus M becomes the group $\tilde{M} = PSL(2,\mathbb{R})$ which is a well-known locally compact group, consisting of all linear fractional transformations $\Phi = (az+b)/(cz+d)$ where $a,b,c,d \in \mathbb{R}$ and $ad - bc = 1$. It is easy to see that G_o becomes the affine group $\tilde{G}_o = \{\psi(w) = aw+b;\ a > 0,\ b \in \mathbb{R}\}$, since these ψ are just the conformal self-maps of $\mathbb{H}$ that leave ∞ fixed. The problem of determining all G_o almost periodic functions is thus equivalent to the problem of determining all $\tilde{G}_o$ almost periodic functions $F \in H^\infty(\mathbb{H})$, where "$\tilde{G}_o$ almost periodic" has a meaning that corresponds in the obvious way to "G_o almost periodic".

Our main theorem says that the only such functions are constants. By contrast, there are many G_{oo} almost periodic functions in $H^\infty(\mathbb{D})$, as one sees by mapping $\mathbb{D}$ onto a horizontal strip so that $z = 1$ goes to $w = +\infty$ and $z = -1$ goes to $w = -\infty$. Then G_{oo} becomes $\tilde{G}_{oo}$, the group of all real translations, and the $\tilde{G}_{oo}$ almost periodic functions are just the analytic functions in the strip that are almost periodic in the usual sense (see Besicovitch, 1954). We take this occasion to acknowledge valuable discussions with I. D. Berg and Robert Kaufman.

Theorem. The only G_o almost periodic functions $f \in H^\infty(\mathbb{D})$ are the constant functions.

Remark. By contrast, there are plenty of non-constant analytic functions that are almost periodic with respect to certain groups described by two or more parameters (see Asisoff, 1965, Petersen, 1938, Tornehave, 1958, Tornehave, 1965). Hence the dimension alone of the group does not seem to be crucial.

Some ideas of a related nature are taken up in Horneffer, 1962, Maak, 1959, and Maak, 1960, the most striking difference being that they consider discrete groups G in contrast to the continuous groups G of the present paper.

Proof of the Theorem. Following the early remarks, it is equivalent to prove the corresponding result for $\tilde{G}_o$. So let $F \in H^\infty(\mathbb{H})$ be $\tilde{G}_o$ almost periodic. We denote by F^* the non-tangential boundary function

$$F^*(x) = \lim_{y\to 0+} F(x+iy), \qquad x \in \mathbb{R}$$

that is associated with F. Now $F^* \in L^\infty(\mathbb{R})$ and we recall that the correspondence $F \leftrightarrow F^*$ sets up an isometric isomorphism between $H^\infty(\mathbb{H})$ and a certain subspace of $L^\infty(\mathbb{R})$ that we call $H^\infty(\mathbb{R})$. Moreover, F is a constant function if and only if F^* is the same constant function.

Now $\mathrm{orb}(F:\tilde{G}_o)$, being precompact and metric, must be separable. Hence $\mathrm{orb}(F^*:\tilde{G}_o)$ is separable in $L^\infty(\mathbb{R})$. We will use this to show that F^* "is uniformly continuous" on $\mathbb{R}$. To do this, we realize $\tilde{G}_o$ geometrically as the upper half-plane $\mathbb{H} = \{b + ia : a > 0,\ b \in \mathbb{R}\}$, and define for $\psi \in \tilde{G}_o$,

$$K(\psi) = F^*(\psi(1))$$

and write

$$K(a,b) = F^*(a+b).$$

We will first prove that $K(\psi) \in L^\infty(\tilde{G}_o)$. Let μ be Haar measure on $\tilde{G}_o$. If E is the set of Lebesgue measure zero in $\mathbb{R}$, off of which F^* is defined and bounded, then $K(a,b)$ is defined for all $(a,b) \notin \tilde{E}$, where

$$\tilde{E} = \{b+ia : a+b \in E\},$$

and we must show that $\mu(\tilde{E}) = 0$. But $d\mu(a,b) = a^{-2}dadb$, as a simple calculation shows, and

$$\mu(\tilde{E}) = \int_{a=0}^{\infty} \int_{b=-\infty}^{\infty} a^{-2}\, \chi_E(a+b)dadb,$$

where χ_E is the characteristic function of E. On changing variables and using Fubini's theorem, we have

$$\mu(\tilde{E}) = \int_{b=-\infty}^{\infty} \int_{t=b}^{\infty} (t-b)^{-2}\, \chi_E(t)dtdb = 0,$$

since E has Lebesgue measure zero. This argument also shows that $K \in L^\infty(\tilde{G}_o)$. Further, it is a simple matter to show that $\mathrm{orb}(K,\tilde{G}_o)$ is separable in $L^\infty(\tilde{G}_o)$ since $\mathrm{orb}(F^*:\tilde{G}_o)$ is separable in $L^\infty(\mathbb{R})$. Now by a theorem of Porta, 1973, from the fact that K has a separable orbit, it follows that K may be redefined on a set of Haar measure zero in $\tilde{G}_o$ to become a uniformly continuous function. Transferring attention to F^*, we see that F^* may be redefined on a set of Lebesgue measure zero to be uniformly continuous on $\mathbb{R}$. This is because $F^*(c) = K(1,c-1)$ and because, in the obvious notation, we have shown that if $\tilde{A}$ has Haar measure zero then A must have Lebesgue measure zero. It might be helpful in this regard to remark that in $\tilde{G}_o$, the map $ax + b$ is close to the map $a'x + b'$ precisely when $b - b'$ is close to 0 and a/a' is close to 1.

Remark. We outline another way to see that F^* is uniformly continuous, via the discussion on pp. 191–192 of Katznelson, 1976. For we observe that $K(\psi)$ is, as an element of $L^\infty(\tilde{G}_o)$, an almost periodic function on $\tilde{G}_o$, so that after redefinition on a set of measure zero, $K(\psi)$ becomes a uniformly continuous function.

Now, in particular, F^* is almost periodic in the usual sense as a function on $\mathbb{R}$. But more importantly, F^* is H almost periodic, where H is the group of homotheties $x \mapsto ax$ $(a > 0)$ of $\mathbb{R}$. We will see that because $\lim_{x\to 0} F^*(x)$ exists, F^* must be constant. For let $F^\#(x) = F^*(e^{-x})$ so that $F^\#$ is an almost periodic function on $\mathbb{R}$ in the usual sense, yet $\lim F^\#(x)$ exists as $x \to +\infty$. Now we may conclude, via the theorem on page 165 of Katznelson, 1976, that $F^\#$ is constant, and thus F^* is constant. For that theorem says that $F^\#$ is the uniform limit of a sequence of trigonometric polynomials $P_n \in W(F^\#)$, where $W(F^\#)$ is the translation convex hull of $F^\#$. Supposing for convenience that $F^\#(x) \to 0$ as $x \to +\infty$, we see that $P(x) \to 0$ as $x \to +\infty$ for all $P \in W(F^\#)$. Since this implies that P (and hence eventually $F^\#$) must vanish identically, the proof is done.

Robert Kaufman has found the following short and simple proof of the main theorem. Suppose $f \in H^\infty(\mathbb{H})$ is $\tilde{G}_o$ almost periodic. Then f^* exists as a non-tangential limit at some $t_o \in \mathbb{R}$. Let

$$\psi_n(w) = \frac{w}{n} + t_o, \quad n = 1,2,3,\dots .$$

Then for every $w \in \mathbb{H}$, $\psi_n(w) \to t_o$ within a sector. Therefore $f(\psi_n(w)) \to f^*(t_o)$ for every $w \in \mathbb{H}$. Since $\mathrm{orb}(f:\tilde{G}_o)$ is pre-compact, we will have $f(\psi_{n_k}(w)) \to f^*(t_o)$ uniformly, for some sequence $n_k \to \infty$. Thus, the range of f on $\mathbb{H}$ is the singleton $\{f^*(t_o)\}$, so that f is a constant, and the result is proved.

In conclusion, it would be interesting to characterize those subgroups G of M that admit G almost periodic bounded analytic functions.

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